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Preface

This book contains the contributions of the XI Congreso del Máster en Investigación y Doctorado en Matemáticas where MSc and PhD students from Universitat Politècnica de València and Universitat de València present their works.

Among the activities carried out by the InvestMat Master, there is the annual Congreso del Máster en Investigación y Doctorado en Matemáticas, which takes place in the Salón de Grados of the Faculty of Mathematics of the Universitat de València.

This congress offers the opportunity for Master and PhD students to present their research work, exchanging ideas with experts in the different research areas and improving their skills when presenting and exhibiting their work in public.

More information about the congress in https://www.uv.es/investmat/info.html.

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Bishop-Phelps-Bollobás Theorem: An Introduction

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1 Introduction

The concepts and questions addressed in this paper fall within the field of Functional Analysis, specifically focusing on norm-attaining operator theory. We will use standard notation, but provide clarification to improve readability: X is a Banach space over the field $\mathbb{K}$, i.e., $\mathbb{R}$ or $\mathbb{C}$ (we will specify which one if necessary). The topological dual of X will be denoted by X^*, while B_X and S_X will be the closed unit ball and the unit sphere of X, respectively. As usual, X^* is endowed with the operator norm,

$$\|f\| = \sup\{|f(x)| : x \in B_X\}, \qquad f \in X^*.$$

At a first glance, it seems natural to ask whether a (continuous linear) functional attains its norm or not. We say that f attain its norm, or it is a norm-attaining functional if the previous supremum is a maximum, i.e., if there exists $x_0 \in B_X$ such that $|f(x_0)| = \|f\|$. The subset of X^* of norm-attaining operators is denoted by $\mathrm{NA}(X)$. Recall the next result, implied by the normed version of the Hahn-Banach Extension Theorem ([11, Theorem 1.9.6]):

Corollary 1.1 *[11, Corollary 1.9.8] Let X be a normed space and $x \in X \setminus \{0\}$, then there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

Consequently, there exist functionals that attain their norm. However, it is not difficult to find functionals that are not in $\mathrm{NA}(X)$:

Example 1.2 *We will show that there exists an element in c_0^* that is not norm-attaining. We start by identifying c_0^* with ℓ_1. Let $f \in c_0^*$ be represented by the sequence $(2^{-n}) \in \ell^1$. Then $\|f\| = 1$. If (a_n) is any element of B_{c_0}, then*

$$|f((a_n))| = \left|\sum_{n=1}^{\infty} 2^{-n} a_n\right| \leq \sum_{n=1}^{\infty} 2^{-n} |a_n| < \sum_{n=1}^{\infty} 2^{-n} = 1,$$

so f does not attain its norm.

It is easy to check that the equality $\mathrm{NA}(X) = X^*$ holds when X is reflexive:

Let f be an element of X^*. Our goal is to show that f is norm-attaining. That is obvious if $f \equiv 0$. If $f \not\equiv 0$, by using Corollary 1.1, there exists $x^{**} \in X^{**}$ such that $\|x^{**}\| = 1$ and

$x^{**}(f) = \|f\|$. As X is reflexive, the canonical embedding $J : X \to X^{**}$ is onto, and there is $x \in X$ such that $J(x) = x^{**}$, then,

$$f(x) = \langle f, J(x) \rangle = \langle f, x^{**} \rangle = \|f\| .$$

Mazur was the first to inquire whether the reciprocal of the previous statement is true, that is, if the equality $\mathrm{NA}(X) = X^*$ implies the reflexivity of X. Although this question was asked in 1933, it was not until 1950 that significant progress was made. Mainly due to James and Klee [8], in 1964, after several years of work, James gave an affirmative response in what is called nowadays James Theorem:

Theorem 1.3 ***(James)*** *[7] Let X be a Banach space. Then X is reflexive if and only if every element of the dual space X^* attain its norm.*

Considering that fact, if X is not reflexive it seems reasonable to ask whether the set of norm-attaining functionals of X is dense in its topological dual. We will discuss this topic in detail in the next section.

2 Bishop-Phelps Theorem

James Theorem states that, if $\mathrm{NA}(X) = X^*$, then X is necessarily reflexive. Keeping that in mind, Phelps [13] named *subreflexive* spaces those normed spaces for which the set of norm-attaining functionals is just dense in X^*.

He started studying the subreflexive spaces in his thesis and discovered that the classical non-reflexive Banach spaces ($c_0, \ell_1, C(X), \ldots$) all were subreflexive, making him wonder if every Banach space must be subreflexive. Phelps, along with Bishop, obtained a positive answer and more general results concerning this problem.

We will start by presenting some technical lemmas, in order to make the rest of the results easier to understand. As most often in Functional Analysis, our starting point will be the real case.

Lemma 2.1 *Let A be a closed subset of a real Banach space X and suppose $f \in S_{X^*}$ is a functional bounded above on A. Then, for each $\gamma \in (0,1)$ and $x \in A$, there exists $x_0 \in A$ such that $x_0 \in x + C$ and $A \cap (x_0 + C) = \{x_0\}$, where C is the closed cone defined as $C = \{x \in X : \gamma \|x\| \leq f(x)\}$.*

A point x_0 satisfying a condition $A \cap (x_0 + C)$ is usually referred to as conical support point of A, where A is a closed subset of a real Banach space and C is a closed cone. Before we state the following result, we recall the following separation theorem:

Theorem 2.2 ***(Hahn-Banach separation Theorem)*** *[2, Theorem 3.26] Let A and B be nonempty convex subsets of a real normed space X such that int$(A) \neq \emptyset$ and int$(A) \cap B = \emptyset$. Then, there exists a nonzero functional $f \in X^*$ that separates A from B. That is*

$$\sup_{y \in B} f(y) \leq \inf_{x \in A} f(x).$$

Moreover, if there is $x_0 \in A \cap B$, then

$$\sup_{y \in B} f(y) = f(x_0) = \inf_{x \in A} f(x).$$

Now, we have the ingredients to present one of the most useful results in this text, which allows us, among other things, to get pretty close to the norm-attaining denseness we were looking for.

Lemma 2.3 *Let A be a closed convex subset of a real Banach space X and suppose $f \in S_{X^*}$ is a functional bounded above on A. If $\varepsilon > 0$ and $x \in A$ are such that*

$$\sup f(A) \leq f(x) + \varepsilon,$$

then, for any $\gamma \in (0,1)$, there exist $g \in X^*$ and $x_0 \in A$ such that

$$\|f - g\| \leq \gamma, \qquad \|x - x_0\| \leq \frac{\varepsilon}{\gamma} \qquad \text{and} \qquad \sup g(A) = g(x_0).$$

Before continuing, it is mandatory to remember some definitions, quite related to the concept of norm-attaining functional.

Definition 2.4 *Let X be a real normed space and A be a convex subset of X. We say a point $x_0 \in A$ is a support point of A if there exists a nonzero functional $f \in X^*$ such that*

$$f(x_0) = \sup_{x \in A} f(x).$$

In that case, we say f is a support functional of A.

The last lemma allows us to present our first Bishop-Phelps Theorem, which we are going to refer to as real Bishop-Phelps support points denseness theorem.

Theorem 2.5 ***(Bishop-Phelps)*** *If A is a closed convex subset of a real Banach space X, then the set of support points of A is dense in the boundary of A.*

Immediately from the definition of support point, we can link the support functionals with the norm-attaining ones as follows:

$$\begin{aligned} f \in \mathrm{NA}(X) &\iff \exists x_0 \in B_X : |f(x_0)| = \|f\| \\ &\iff \exists x_0 \in B_X : f(x_0) = \sup_{x \in B_X} f(x) \\ &\iff f \text{ is a support functional of } B_X. \end{aligned}$$

The second equivalence is due to $\|f\| = \sup_{x \in B_X} |f(x)| = \sup_{x \in B_X} f(x)$. The norm-attaining functionals of X are exactly the support functionals of the closed unit ball of X.

Theorem 2.6 ***(Bishop-Phelps)*** *Let X be a real Banach space and $A \subset X$ be closed and convex. If $f \in S_{X^*}$ is bounded above on A and $\delta \in (0,1)$, then there exists $g \in X^*$, a support functional of A, with $\|f - g\| < \delta$.*

The following corollary is commonly known as real Bishop-Phelps support functional denseness theorem.

Corollary 2.7 *If X is a real Banach space and $A \subset X$ is bounded, closed and convex, then the set of support functionals of A is dense in X^*.*

The next corollary is the original Bishop-Phelps Theorem that was proved in their 1961 article [5]. As is common in mathematics, improvements were made to the result while preserving the same idea and the new results were still called Bishop-Phelps Theorems, as is the case with the previous Corollary and both of the theorems presented in this section.

Corollary 2.8 *Every real Banach space is subreflexive.*

3 Extensions of the Bishop-Phelps Theorem

3.1 Complex Bishop-Phelps Theorem

The first question the reader should have in mind concerns the complex case of the Bishop-Phelps Theorem. We need to pay special attention about the definitions we provide, as they entirely determine whether the theorem holds in the complex case or not.

Definition 3.1 *Let X be a complex Banach space and A be a convex subset of X. We say $x_0 \in A$ is a support point of A if there exists a nonzero functional $f \in X^*$ such that*

$$\mathrm{Re} f(x_0) = \sup_{x \in A} \mathrm{Re} f(x).$$

In that case, we say f is a support functional of A. On the other hand, $x_0 \in A$ is a modulus-support point if there exists a nonzero functional $f \in X^*$ such that

$$|f(x_0)| = \sup_{x \in A} |f(x)| .$$

In this case, we say f is a modulus-support functional of A.

Proposition 3.2 *Let X be a complex Banach space and A be a convex subset of X. If A is balanced and $x_0 \in A$ is a support point of A (respectively support functional), then x_0 is a modulus-support point of A (respectively modulus-support functional), and so an element of ∂A.*

Proposition 3.3 *[11, Proposition 1.9.3] Let X be a complex normed space and $X_{\mathbb{R}}$ the real normed space obtained by restricting the multiplication of vectors by scalars to $\mathbb{R} \times X$. The following statements are true.*

1. *If $f \in X^*$, then $u = \mathrm{Re} f \in (X_{\mathbb{R}})^*$ and, for all $x \in X$,*

$$f(x) = u(x) - iu(ix).$$

2. *If $u \in (X_{\mathbb{R}})^*$ and we define f by the formula above, then $f \in X^*$ and $u = \mathrm{Re} f$.*
3. *If $f \in X^*$ and $u = \mathrm{Re} f$, then $\|f\| = \|u\|$.*

These basic relations between a complex normed space, its corresponding real normed space, a functional and its real part are the elemental tool to study the complex case of Bishop-Phelps Theorems. The next theorem is the complex Bishop-Phelps support points denseness theorem.

Theorem 3.4 ***(Bishop-Phelps)*** *Let X be a complex Banach space and A be a closed, convex subset of X. Then, the set of support points of A is dense in the boundary of A.*

If we denote the set of support points of a convex set A by $\mathrm{SP}(A)$ and the set of modulus-support points of A by $\mathrm{MSP}(A)$, the Proposition 3.2 tells us that, if A is balanced, then $\mathrm{SP}(A) \subset \mathrm{MSP}(A) \subset \partial A$. The complex version of the Bishop-Phelps Theorem for support points guarantees that $\mathrm{SP}(A)$ is dense in ∂A and so is $\mathrm{MSP}(A)$. Therefore we have the following corollary.

Corollary 3.5 *Let X be a complex Banach space and A be a closed, convex subset of X. If A is balanced, the set of modulus-support points of A is dense in the boundary of A.*

We do now present the complex analogous of the Corollary 2.7, called the complex Bishop-Phelps functional denseness theorem.

Theorem 3.6 ***(Bishop-Phelps)*** *Let X be a complex Banach space and A be a bounded, closed and convex subset of X. Then the set of support functionals of A is dense in X^*.*

Similarly to the real case, the set of the modulus-support functionals of the closed unit ball of X is exactly the set of norm-attaining operators of X:

$$\begin{aligned} f \in \mathrm{NA}(X) &\iff \exists x_0 \in B_X : |f(x_0)| = \sup_{x \in B_X} |f(x)| \\ &\iff f \text{ is a modulus-support functional of } B_X. \end{aligned}$$

Corollary 3.7 *Every complex Banach space is subreflexive.*

The classical result of Bishop and Phelps is now at hand:

Theorem 3.8 ***(Bishop-Phelps)*** *Every Banach space (real or complex) is subreflexive.*

Next, we show one of the main differences between the complex and real cases of Bishop-Phelps theorems. Paraphrasing Phels, "At a 1985 conference at Kent State University, Godefroy asked if the set of modulus-support functionals of a bounded, closed, convex set of a complex Banach space is dense in the dual space". While the question has a positive answer in the real case, is not true for complex Banach spaces.

Theorem 3.9 *Let X be a real Banach space and let A be a bounded, closed, convex subset of X. Then, the set of modulus-support functionals of A is dense in X^*.*

The question of Godefroy reimaned open until 2000, when Lomonosov [10] proved that there is a complex Banach space X such that X^* is $\mathcal{H}^\infty$, the space of analytic bounded functions on the open unit disk, such that there exists A a nonempty, closed, convex and bounded of X such that the set of modulus-support functionals of A is empty, and so it is impossible for it to be dense, giving a negative answer to the Godefroy question.

3.2 Does X need to be Banach?

If we ask about the Banach condition to inquire about the denseness of the set of norm-attaining functionals, the answer to is an absolute yes. On the one hand, if X is still a normed space but is not a complete one, Phelps [13] constructed an incomplete normed space F with the following properties:

1. F is not subreflexive.
2. F is isomorphic to a subreflexive space F'.

If we ask about the denseness of the set of support functionals of a bounded, closed and convex set the answer is no, even if we preserve the completeness but we weaken the structure, for example, considering a Frèchet space (locally convex, metrizable and complete). Peck [12] proved that if E a Frèchet spaces obtained as the product space of an infinite sequence of non-reflexive Banach spaces, then E contains a bounded, closed, convex subset without support functionals.

3.3 Bishop-Phelps Theorem for operators

Considering that $\mathrm{NA}(X) = \mathrm{NA}(X, \mathbb{K})$, that $X^* = B(X, \mathbb{K})$ and that $\mathbb{K}$ is a Banach space, a feasible question is, what happens if we substitute $\mathbb{K}$ with Y, another Banach space over $\mathbb{K}$? Do we still have denseness of $\mathrm{NA}(X, Y)$ in $B(X, Y)$?

The definition of a norm-attaining operator is clear: an operator $T \in B(X, Y)$ is an element of $\mathrm{NA}(X, Y)$ if there exists $x_0 \in X$ such that $\|T(x_0)\| = \|T\|$.

Lindenstrauss [9] was the first to provide a negative answer, by demonstrating that, if X is a strictly convex Banach space and there is a non-compact operator from c_0 to X, then the subset of norm-attaining operators from c_0 to X is not dense in $B(c_0, X)$. As a consequence, if X is strictly convex space isomorphic to c_0, then the subset $\mathrm{NA}(c_0, X)$ is not dense in $B(c_0, X)$.

4 Bishop-Phelps-Bollobás Theorem

4.1 Bollobás' Theorem

We now introduce Bollobás, whose theorem not only establishes the density of norm-attaining functionals but also provides simultaneous control over points and functionals. The theorems of Bishop-Phelps (Theorem 3.8) and Bollobás present slightly different statements, yet it is noteworthy that the Bishop-Phelps theorem is a particular case of the Bollobás theorem, as mentioned earlier.

Theorem 4.1 ***(Bollobás' theorem)****[4, Theorem 1] Let X be a Banach space. Suppose that $x \in S_X$ and $f \in S_{X^*}$ holding*

$$|f(x) - 1| \leq \frac{\varepsilon^2}{2}, \quad 0 < \varepsilon < \frac{1}{2}.$$

Then there are $y \in S_X$ and $g \in S_{X^*}$ such that

$$g(y) = 1, \quad \|y - x\| < \varepsilon + \varepsilon^2, \quad \|g - f\| \leq \varepsilon.$$

Bollobás' Theorem is the best possible result in the next sense. For any $\varepsilon \in (0,1)$ there exist a Banach space E such that a point $x \in S_E$ and a functional $f \in S_{E^*}$ such that $f(x) = 1-(\varepsilon^2/2)$. If we consider $y \in S_E$ and $g \in S_{E^*}$ with $g(y) = 1$ then either $\|f - g\| \geq \varepsilon$ or $\|x - y\| \geq \varepsilon$.

We are going to see this with an example. Consider $E = \mathbb{R}^2$ Banach space and lets take the following set as the unit ball

$$\{(a, b) \in \mathbb{R}^2 : -1 \leq a + (1 - \varepsilon)b \leq 1, -1 \leq b \leq 1\}.$$

Let f the following functional

$$f(a, b) = a\frac{\varepsilon}{2} + \left(1 - \frac{\varepsilon^2}{2}\right) b.$$

If $x = (0, 1)$ then $f(x) = 1 - \frac{\varepsilon^2}{2}$. Besides, we get that $\|f\| = 1$ since $\varepsilon \in (0, 1)$. If $g \in S_{E^*}$ such that $\|f - g\| < \varepsilon$ then we can say that g attain its the supremum at the same point as f, that is, at the point $(\varepsilon, 1)$. But $\|(\varepsilon, 1) - x\| = \|(\varepsilon, 0)\| = \varepsilon$.

4.2 Bishop-Phelps-Bollobás' Theorem

From Bollobás' original result we can obtain the sharpest version that is what we will refer to as Bishop-Phelps-Bollobás Theorem. To achieve this, we will introduce the concept of Bishop-Phelps-Bollobás moduli of a Banach space.

Definition 4.2 *Let X be a real or complex Banach space and let $\delta > 0$. We define the* ***Bishop-Phelps-Bollobás modulus of*** *X as the function $\Phi_X : (0,2) \to \mathbb{R}^+$ such that given a $\delta \in (0,2)$ then $\Phi_X(\delta)$ is the infimum of those $\varepsilon > 0$ such that for every $(x,f) \in B_X \times B_{X^*}$ with $\mathrm{Re} f(x) > 1 - \delta$ there is $(y,g) \in \Pi(X)$ such that $\|x-y\| < \varepsilon$ and $\|f-g\| < \varepsilon$.*

Here $\Pi(X)$ is the following set

$$\{(x,f) \in X \times X^* : \|x\| = \|f\| = f(x) = 1\}.$$

Let us also define the **spherical Bishop-Phelps-Bollobás modulus**, $\Phi_X^S(\delta)$, as the previous one but instead of take $(x,f) \in B_X \times B_{X^*}$ for the spherical one we take $(x,f) \in S_X \times S_{X^*}$. Its easy to see that $\Phi_X^S(\delta) \le \Phi_X(\delta)$, so any estimation from above $\Phi_X(\delta)$ is valid for $\Phi_X^S(\delta)$ and viceversa.

Theorem 4.3 *[6, Theorem 2.1] For every Banach space X and every $\delta \in (0,2)$, we have that $\Phi_X(\delta) \le \sqrt{2\delta}$, and so $\Phi_X^S(\delta) \le \sqrt{2\delta}$.*

This result let us rewrite Bollobás theorem in their sharpest version as the following theorem.

Theorem 4.4 ***(Bishop-Phelps-Bóllobas Theorem****[6, Corollary 2.4] Let X be a Banach space. Let $\varepsilon \in (0,2)$ and suppose that $x \in B_X$ and $f \in B_{X^*}$ satisfying*

$$\mathrm{Re} f(x) > 1 - \frac{\varepsilon^2}{2}.$$

Then there exist $y \in S_X$ and $g \in S_{X^*}$ such that

$$g(y) = 1, \quad \|y - x\| < \varepsilon, \quad \|g - f\| < \varepsilon.$$

4.3 Bishop-Phelps' result as a particular case of Bollobás' Theorem

Before end this section we should comment that we can deduce Bishop-Phelps Theorem from Bollobás Theorem.

Given $f \in S_{X^*}$, we want to prove there exists $g \in S_{X^*}$ such that it attains its norm and is close to f. Since $\|f\| = 1$, we can choose $x \in S_X$ such that $|f(x)| > 1 - \delta$ with $\delta > 0$. Without losing generality, we can take $\delta = \frac{\varepsilon^2}{2}$ with $0 < \varepsilon < \frac{1}{2}$. Applying Bollobás' Theorem, there exist $y \in S_X$ and $g \in S_{X^*}$ such that $g(y) = 1$, so it attains its norm, $\|x - y\| < \varepsilon + \varepsilon^2$, and $\|f - g\| \le \varepsilon$. The last inequality gives us the density of $\mathrm{NA}(X)$ in X^*, so we obtain the Bishop-Phelps Theorem.

5 Bishop-Phelps-Bollobás Property

Despite having a counterexample regarding the density of $\mathrm{NA}(X,Y)$ in $B(X,Y)$ from Section 3.3, it is possible to apply the Bishop-Phelps-Bollobás Theorem to the space $B(X,Y)$? Unfortunately, the answer is a solid no. Lindenstrauss not only provided a counterexample but also formulated a proposition regarding the density of $\mathrm{NA}(X,Y)$ in $B(X,Y)$.

Proposition 5.1 *[9, Proposition 5] There exist a Banach spaces X for which* $\mathrm{NA}(X,X)$ *is not dense in $B(X,Y)$.*

With this result, we understand that extending Bishop-Phelps Theorem to operators is not possible, implying that deducing a Bishop-Phelps-Bollobás Theorem for operators is also not possible. But, let us reformulate the initial question: is it possible to have a version of the Bishop-Phelps-Bollobás Theorem for operators? The answer to this question is provided by the following definition.

Definition 5.2 *Let X and Y be Banach spaces. We say that the pair (X,Y) have the* ***Bishop-Phelps-Bollobás property for operators (BPBp)*** *if given $\varepsilon > 0$, there exist a $\eta(\varepsilon) > 0$ such that whenever $T \in B(X,Y)$ with $\|T\| = 1$ and $x \in S_x$ holds*

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exist $S \in B(X,Y)$ with $\|S\| = 1$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x - x_0\| < \varepsilon, \quad \|S - T\| < \varepsilon.$$

Some authors require the existence of a $\beta(\varepsilon) > 0$ such that $\lim_{t\to 0^+} \beta(t) = 0$, so that instead of having $\|x - x_0\| < \varepsilon$, we would actually have $\|x - x_0\| < \beta(\varepsilon)$. However, for this article, we will not consider this requirement for the definition.

To clarify, the property does not hold in general. If we consider the space ℓ_1^2 as the 2-dimensional space $(\mathbb{R}^2, \|\cdot\|_1)$, then there exists a Banach space Z such that (ℓ_1^2, Z) fails BPBp. The proof can be found in [3, Example 4.1], as the construction of Z is non-trivial.

5.1 BPBp for classical Banach spaces

Our objective now is to provide some results to determine for which Banach spaces the BPBp holds. By classical Banach spaces, we mean those that students encounter during their undergraduate or even master's degree studies. We will only state the results, as their proofs are extensive for this article.

Theorem 5.3 *[1, Proposition 2.4] Let X and Y be finite dimensional Banach spaces. Then the pair (X,Y) have the BPBp.*

Many space properties are related to BPBp. We will only state one, as the results involve classical Banach spaces.

Definition 5.4 *A Banach space X has the* ***approximate hyperplane series property (AHSp)*** *if for every $\varepsilon > 0$ there exist $\gamma(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\lim_{t\to 0^+} \gamma(t) = 0$ such that for every sequence $(x_k) \subset S_X$ (or in B_X) and every convex series $\sum_{k\geq 1} \alpha_k$ satisfying*

$$\left\| \sum_{k=1}^{+\infty} \alpha_k x_k \right\| > 1 - \eta(\varepsilon),$$

there exist a subset $D \subset \mathbb{N}$, $\{z_k : k \in D\} \subset S_X$ and $f \in S_{X^*}$, such that

1. *$\sum_{k\in D} \alpha_k > 1 - \gamma(\varepsilon)$.*
2. *$\|z_k - x_k\| < \varepsilon$ for all $k \in D$.*
3. *$f(z_k) = 1$ for all $k \in D$.*

Proposition 5.5 *[1, Proposition 3.5] Every finite-dimensional Banach space has AHSp.*

Proposition 5.6 *[1, Proposition 3.4] For every σ-finite measure μ, the space $L^1(\mu)$ (real or complex) has AHSp.*

Proposition 5.7 *[1, Proposition 3.7] The real or complex spaces $C(K)$ have AHSp for any compact Hausdorff space K.*

From [1, Proposition 3.7] proof we can obtain a corollary for the Banach space of continuous functions on Ω that vanishes at ∞, denoted by $C_0(\Omega)$.

Corollary 5.8 *The real or complex spaces $C_0(\Omega)$ have AHSp for any locally compact space Ω.*

We recall the concept of a **uniformly convex space**. A normed space X is uniformly convex if for every $\varepsilon > 0$ there exist a $0 < \delta < 1$ such that for all $x, y \in B_X$ such that $\frac{\|x+y\|}{2} > 1 - \delta$, we have $\|x - y\| < \varepsilon$.

An example of uniformly convex Banach spaces includes $L^p(\mu)$ for $1 < p < \infty$ and for every σ-finite measure μ.

Proposition 5.9 *[1, Proposition 3.8] A uniformly convex Banach space has AHSp.*

Theorem 5.10 *[1, Theorem 4.1] A Banach space Y is such that the couple (ℓ_1, Y) has the BPBp if, and only if, Y satisfies AHSp.*

We can summarize these results with the following tables.

Table 1.1: Classical Banach spaces with BPBp (I)

BPBp	$\forall Y$	$\mathbb{K}$	F.D.	c_0	ℓ_1	ℓ_q	ℓ_∞
$\forall X$	✗	✓		✓	✗	✗	✓
$\mathbb{K}$	✓	✓	✓	✓	✓	✓	✓
F.D.	✗	✓	✓	✓			✓
c_0	✗	✓		✓	$\checkmark_{\mathbb{C}}$	✓	✓
ℓ_1	✗	✓	✓	✓	✓	✓	✓
ℓ_p	✓	✓	✓	✓	✓	✓	✓
ℓ_∞	✗	✓		✓	$\checkmark_{\mathbb{C}}$	✓	✓
$L^1(\mu)$	✗	✓	$\checkmark_{\sigma}$	✓	✓	✓	✓
$L^p(\mu)$	✓	✓	✓	✓	✓	✓	✓
$L^\infty(\mu)$	✗	✓		✓	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R}\vert\mathbb{C}}$	✓
$C(K_1)$	✗	✓		✓	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R}\vert\mathbb{C}}$	✓
$C_0(L_1)$	✗	✓		✓	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R}}$	✓

Table 1.2: Classical Banach spaces with BPBp (II)

BPBp	$L^1(\nu)$	$L^q(\nu)$	$L^\infty(\nu)$	$C(K_2)$	$C_0(L_2)$
$\forall X$	✗	✗		✗	✗
$\mathbb{K}$	✓	✓	✓	✓	✓
F.D.			✓	✓	✓
c_0	$\checkmark_{\mathbb{C}}$	✓	✓	✓	✓
ℓ_1	✓	✓	✓	✓	✓
ℓ_p	✓	✓	✓	✓	✓
ℓ_∞	$\checkmark_{\mathbb{C}}$	✓	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	✓
$L^1(\mu)$	✓	✓	$\checkmark_{2_\circ}$		
$L^p(\mu)$	✓	✓	✓	✓	✓
$L^\infty(\mu)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R}\vert\mathbb{C}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	
$C(K_1)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R}\vert\mathbb{C}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}\vert 1_m}$
$C_0(L_1)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R}\vert 1_m}$	$\checkmark_{\mathbb{R}\vert 1_m}$	$\checkmark_{\mathbb{R}\vert 1_m}$

Table 1 and 2 use the following notation.

- $1 < p, q < \infty$, μ, ν are measures, K_1, K_2 any compact Hausdorff space, L_1, L_2 are any locally compact Hausdorff spaces, and F.D. denote the finite dimensional Banach spaces.
- The symbol ✓ means that the pair has the BPBp in general, and possibly under some extra conditions.
- ✗ means that there is at least 1 known counterexample.
- The blank space indicates that no answer is currently known.
- We use the subindex $\mathbb{R}$ to refer to the real case, likewise subindex $\mathbb{C}$ to the complex case.
- We will use 1_{subindex} to refer that whatever comes next applies to the domain space, and 2_{subindex} for the range space.
- σ stands for σ-finite measure.
- $\circ$ stands for localizable measure.
- m means that the corresponding (locally) compact Hausdorff spaces is metrizable.

5.2 Open questions

Regardless, the blank spaces raise some questions, and there are still more for which we don't know the answers yet. We will present just a few of them, those ones that we consider more important or interesting.

Question 1: Provide a *direct* proof to the fact that the pair (X, Y) have BPBp if both are finite-dimensional Banach spaces.

From Theorem 5.3, we know that the statement is true, but the proof relies on a contradiction using the compactness of the unit ball in both spaces.

Question 2: Is it true that all finite-rank operators can be approximated by norm-attaining ones?

It is not even known whether the answer is true when we consider $\mathbb{R}^2$ endowed with the Euclidean norm.

Question 3: It is true that the pair (c_0, ℓ_1) have the BPBp in the real case?

While we know that the complex case is true, we cannot make any assertions about the real one.

Question 4: Let X be a uniformly convex space. We denote $(X, \ldots, X; \mathbb{K})$ as the space of N-linear functions over the N-th cartesian product of X. It is true that $(X, \ldots, X; \mathbb{K})$ have BPBp for symmetric N-linear forms?

Recall that an $F \in (X, \ldots, X; \mathbb{K})$ is symmetric if $F(x_1, \ldots, x_N)$ is invariant under any permutation of x_i with $1 \leq i \leq N$. This question has a positive answer in the case of symmetric bilinear forms on Hilbert spaces.

Question 5: It is true that $\Phi_X^S(X, \mathbb{R}, \eta) \leq \min\{\sqrt{2\eta}, 1\}$ for every Banach space X?

In the same way we define the Bishop-Phelps-Bollobás module for a Banach space, we can do so for a pair of Banach spaces (X, Y). Without delving into further definitions, the unique difference between $\Phi_X^S(X, \mathbb{R}, \eta)$ and $\Phi_X^S(\eta)$ is that in the first one, we consider $|g(y)| = 1$ instead of the second one, where we simply take $g(y) = 1$.

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Aplicación de transformaciones en el dominio de sinograma para el aumentado de datos

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1. Introducción

La tomografía por emisión de positrones (en inglés, PET) es una técnica de adquisición de imagen médica, basada en la detección de dos fotones de una energía de 511 KeV, provenientes de la un proceso de aniquilación de un electrón, con su antipartícula, el positrón proveniente de un radiotrazador inyectado en el paciente o objeto de estudio. La detección de estos fotones forman una línea recta con un ángulo de 180° entre ambos fotones, debido al proceso de aniquilación. Esta línea que se genera se le conoce como la línea de respuesta (en inglés, Line of response o LOR) que se puede proyectar del espacio de la imagen real a un nuevo dominio, denominado espacio sinograma. Este espacio sinograma se define por dos valores S(r,θ) donde la r define la distancia del centro del escáner a la LOR y el ángulo θ que define dicha LOR con el eje del escáner. Los valores de estos datos en el espacio sinograma se pueden agrupar para obtener un histograma 2D de todos los procesos de coincidencia de la detección. Estos histogramas son lo que denominamos un sinograma, las imágenes en este dominio no son interpretables por las personas, por lo que deberemos aplicar algoritmos de reconstrucción para obtener la representación real 3D del objeto utilizado durante la detección. Uno de los algoritmos de reconstrucción más empleados es un método iterativo denominado maximización de la expectación de máxima verosimilitud (en inglés, Maximum -Likelihood Expectation Maximization o MLEM) que nos permite a partir de la imagen en el espacio sinograma obtener la imagen del objeto real empleado durante el proceso de detección.

En el estado del arte de la reconstrucción de imagen PET se están obteniendo resultados muy positivos haciendo uso de redes neuronales convolucionales (en inglés, Convolutional Neural Networks o CNNs) para la reconstrucción de imágenes PET a partir de las imágenes en el espacio sinograma. Las redes neuronales necesitan de grandes cantidades de datos durante la fase del entrenamiento de la red. Con esta finalidad existen una serie de métodos denominados técnicas de aumentado de datos que nos permiten de forma sintética, partiendo de las imágenes originales, obtener una mayor cantidad de imágenes

[1],[2]. A la hora de aplicar los algoritmos de aumentado de datos, estos se aplican directamente sobre la imagen real final reconstruida después de la adquisición. Para posteriormente aplicar una transformada de Radon[7], que nos permite obtener una imagen en el espacio sinograma a partir de una imagen real y obtener de esta forma un sinograma transformado que posteriormente se utiliza para el entrenamiento de una la red neuronal que permita la reconstrucción de la imagen en el dominio real partiendo de la imagen del dominio de sinograma.

En este articulo exploramos la posibilidad de aplicar los métodos directamente sobre el espacio sinograma, en lugar de tener que aplicar las transformaciones sobre la imagen real final, para después poder obtener el sinograma con las transformaciones pertinentes, con la finalidad de obtener un mayor conjunto de datos para poder aplicar estos en un posterior desarrollo de una red neuronal para la reconstrucciones de imágenes PET.

2. Metodología

Con el fin de comprobar nuestras asumpciones se ha partido de las ecuaciones transformación de espacio real al espacio sinograma, donde se ha tomado la notación de la convención sentido horario para definir los valores de r y θ:

$$r = \frac{x_1(y_1 - y_2) + y_1(x_2 - x_1)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}} = \frac{-x_1 y_2 + x_2 y_1}{\sqrt{(x_1 - x_2)^2 - (y_1 - y_2)^2}} \tag{1.1}$$

$$\theta = \arctan(x_1 - x_2, y_1 - y_2) = \begin{cases} \arctan(\frac{x_1 - x_2}{y_1 - y_2}) & \text{si } x_1 - x_2 > 0 \\ \frac{\pi}{2} - \arctan(\frac{y_1 - y_2}{x_1 - x_2}) & \text{si } y_1 - y_2 > 0 \\ -\frac{\pi}{2} + \arctan(\frac{x_1 - x_2}{y_1 - y_2}) & \text{si } y_1 - y_2 < 0 \\ -\arctan(\frac{x_1 - x_2}{y_1 - y_2}) \pm \pi & \text{si } x_1 - x_2 < 0 \\ \text{Indet} & \text{si } x_1 = x_2 = 0, \\ & \text{si } y_1 = y_2 = 0 \end{cases} \tag{1.2}$$

Tomando estas ecuaciones vamos a explorar cómo podríamos aplicar las técnicas de transformación para el aumentado de datos directamente en el espacio de sinograma y estudiar estas transformaciones sobre las ecuaciones y las implicaciones que tendrían sobre las imágenes del espacio sinograma.

2.1. Rotaciones

Considerando las ecuaciones (1.1) y (1.2), así como la siguiente matriz de rotación, donde ϕ es el ángulo de rotación aplicado al sistema en el espacio de la imagen real:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{matrix} x' = x\cos\phi - y\sin\phi \\ y' = x\sin\phi + y\cos\phi \end{matrix}$$

Realizando las sustituciones necesarias llegamos a las siguientes expresiones para la aplicación de rotaciones:

$$r^{rotado} = \frac{-x_1 y_2 + y_1 x_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}} = r$$

Por tanto

$$\theta^{rotado} = \theta - \phi = \arctan(x_1' - x_2', y_1' - y_2') = \begin{cases} \arctan(\frac{x_1 - x_2}{y_1 - y_2}) - \phi & \text{si } x_1 - x_2 > 0 \\ \frac{\pi}{2} + \arctan(\frac{y_1 - y_2}{x_1 - x_2}) - \phi & \text{si } y_1 - y_2 > 0 \\ -\frac{\pi}{2} + \arctan(\frac{y_1 - y_2}{x_1 - x_2}) - \phi & \text{si } y_1 - y_2 < 0 \\ \arctan(\frac{x_1 - x_2}{y_1 - y_2}) - \phi \pm \pi & \text{si } x_1 - x_2 < 0 \\ \text{Indet} & \text{si } x_1 = x_2 = 0, \\ & \text{si } y_1 = y_2 = 0 \end{cases}$$

Por lo tanto concluimos que las transformaciones de tipo rotaciones directamente sobre el espacio sinograma quedan resumidas de la siguiente manera:

$r^{rotado} = r$ es invariante
$\theta^{rotado} = \theta - \phi$ con ϕ en ángulo de rotación.

Considerando esta base matemática, se ha desarrollado el siguiente algoritmo para implementar las rotaciones en el espacio sinograma:

Algorithm 1 Rotación imagen en el dominio sinograma

Definiciones
S_{datos} := Array que contiene la r y θ a partir de un conjunto de datos de una adquisición PET
dim_S := Dimensiones del array S_{datos}
pix_{desp} := Número de píxeles a desplazar la imagen
Img_desp := imagen final del sinograma desplazada
Argumentos
S_{datos}
pix_{desp}
$media\_ultimos\_pixeles = True$
Inicio
$dim_S \leftarrow$ Dimensiones(S_{datos})
$Img\_desp \leftarrow$ ceros(dim_S)
$pix\_rest \leftarrow dim_S - pix_{desp}$
$imagen\_recortada \leftarrow S_{datos}[:, : pix_{desp}]$
Img_desp[:,pix_rest:dim_S] $\leftarrow$ Inversión(S_{datos},0) ▷ Array 2D invertido en la eje X
Img_desp[:,:pix_rest] $\leftarrow S_{datos}$[:,pix_{desp}:dim_S]
if $media\_ultimos\_pixeles = True$ **then**
    Img_desp[:,pix_rest] $\leftarrow (Img\_desp[:, pix\_rest + 1] + Img\_desp[:, pix\_rest - 1])/2$
end if
Devuelve Img_desp

2.2. Escalado

Considerando de nuevo las ecuaciones (1.1) y (1.2), así como las siguientes transformaciones, donde λ es factor de escalado :

$$x^{escalado} = \lambda x$$
$$y^{escalado} = \lambda y$$

Sustituyendo sobre las ecuaciones obtenemos que:

$$r^{escalado} = \lambda r$$

Y adicionalmente

$$\theta^{escalado} = \theta = \arctan(x_1' - x_2', y_1' - y_2') = \begin{cases} \arctan(\frac{x_1 - x_2}{y_1 - y_2}) & \text{si } x_1 - x_2 > 0 \\ \frac{\pi}{2} + \arctan(\frac{y_1 - y_2}{x_1 - x_2}) & \text{si } y_1 - y_2 > 0 \\ -\frac{\pi}{2} + \arctan(\frac{y_1 - y_2}{x_1 - x_2}) & \text{si } y_1, y_2 < 0 \\ \arctan(\frac{x_1 - x_2}{y_1 - y_2}) \pm \pi & \text{si } y_1 - y_2 < 0 \\ \text{Indet} & \text{si } x_1 = x_2 = 0, \\ & \text{si } y_1 = y_2 = 0 \end{cases}$$

Por lo tanto concluimos que las transformaciones de tipo escalado directamente sobre el espacio sinogramas quedan de la siguiente manera:

$$\boxed{\begin{array}{l} r^{escalado} = \lambda r \\ \theta^{escalado} = \theta \text{ es invariante.} \end{array}}$$

Considerado lo anterior se ha desarrollado el siguiente algoritmo:

Algorithm 2 Transformación de escalado

Definiciones
$Imagen$:= Array de la imagen original
$Imagen_{dim}$:= Dimensiones de la imagen original
$Zoom$:= Función zoom de SciPy
$Factor_{escalado}$:= Factor de escalado
$Imagen\_escalada$:= Array de la imagen escalada
$Imagen\_escalada_{dim}$:= Dimensiones de la imagen una vez escalada
$Imagen\_escalada_{index}$:= Índice de inicio de la imagen escalada
$Imagen\_escalada_{index\_end}$:= Índice de fin de la imagen escalada
$Imagen\_final$:= Array de la imagen escalada, ajustada al rango de la imagen original
Argumentos
$Imagen$
$Factor_{escalado}$
Inicio
$Imagen\_escalada \leftarrow$ Escalado(Imagen,($Factor_{escalado}$, 1))
$Imagen_{dim} \leftarrow$ Dimensiones(Imagen)
if $Factor_{escalado} < 0$ **then**
    $Imagen\_escalada_{dim} \leftarrow$ Dimensiones(Imagen_escalada)
    $Imagen\_escalada_{index} \leftarrow Parte\_entera(Imagen_{dim} - Imagen\_escalada_{dim})$
    $Imagen\_escalada_{index\_end} \leftarrow (Imagen\_escalada_{dim} + Imagen\_escalada_{index})$
    Imagen_final $\leftarrow$ Ceros($Imagen_{dim}$)
    Imagen_final $\leftarrow$ Imagen_escalada($Imagen\_escalada_{index}$:$Imagen\_escalada_{index\_end}$,:)
end if
if $Factor_{escalado} > 0$ **then**
    $Imagen\_escalada_{dim} \leftarrow$ Dimensiones(Imagen_escalada)
    $Imagen\_escalada_{index} \leftarrow |Parte\_entera(Imagen_{dim} - Imagen\_escalada_{dim})|$
    $Imagen\_escalada_{index\_end} \leftarrow (Imagen\_escalada_{dim} + Imagen\_escalada_{index})$

Imagen_final $\leftarrow$ Ceros($Imagen_{dim}$)
Imagen_final $\leftarrow$ Imagen_escalada($Imagen\_escalada_{index}$:$Imagen\_escalada_{index\_end}$,:)
end if
if $Factor_{escalado} == 0|Factor_{escalado} == 1$ **then**
$Imagen\_final == Imagen$
end if
Devuelve: $Imagen\_final$

3. Experimentos

Para verificar la validez de los resultados, decidimos tomar dos aproximaciones diferentes.

- Transformaciones sobre los datos en modo lista.
  1. Partimos del histograma 2D de la imagen en el domino de sinograma.
  2. Aplicamos el algoritmo de transformación en el dominio de sinograma.
  3. Reconstruimos aplicando la transformada inversa de Radon, para obtener la imagen en el dominio real.
- Transformaciones sobre la imagen original.
  - Método 1 - Conversión a sinograma y transformación.
    1. Partimos de la imagen en el dominio real y aplicamos la transformada de Radon para obtener el sinograma.
    2. Aplicamos el algoritmo de transformación en el espacio sinograma.
    3. Reconstruimos aplicando la transformada inversa de Radon, para obtener la imagen en el dominio real.
  - Método 2 - Transformación y conversión a sinograma.
    1. Partimos de la imagen en el dominio real y aplicamos la transformación.
    2. Aplicamos la transformada de Radon para obtener la imagen en el espacio sinograma.
    3. Reconstruimos aplicando la transformada inversa de Radon, para obtener la imagen en el dominio real.

Por un lado transformamos las imágenes originales primero y luego aplicamos la transformación al espacio sinograma y por otro lado convertimos la imagen original a sinograma y luego aplicamos el algoritmo desarrollado. Por otro lado, se aplica el algoritmo implementado directamente sobre los datos de la adquisición y tras esto convertimos los datos a un histograma 2D.

Para ello se han implementado los algoritmos comentados en la sección anterior y se han aplicado.

3.1. Rotaciones

En la figura 1.1 podemos ver los resultados de la aplicación del algoritmo de rotación desarrollado y su comparación para los diferentes métodos empleados. En este caso, se ha tomado el valor de *media_ultimos_pixeles*, como verdadero. Para corregir los artefactos que aparecen por el hecho de rotar las imágenes directamente.

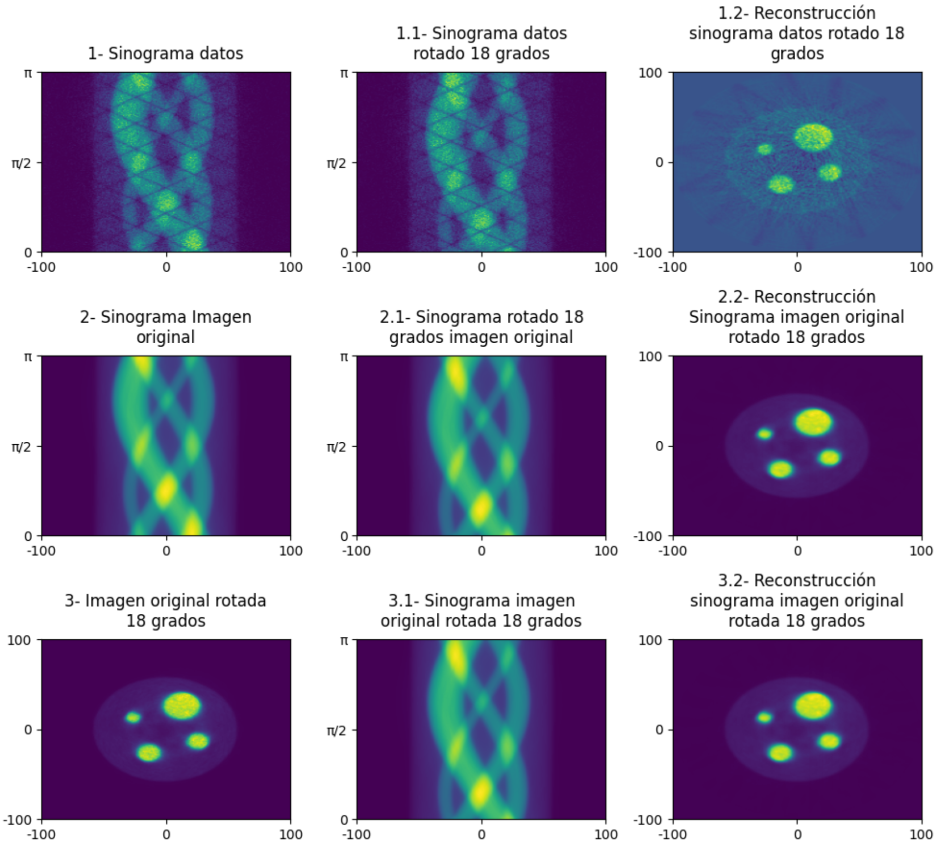

Figura 1.1: Resultados de aplicar una rotación de 18 °

3.2. Escalado

En la figura 1.2 y 1.3 podemos ver los resultados del algoritmo de escalado desarrollado. En este caso se ha comparado los resultados para un escalado de alejamiento de 0.5 y otro de acercamiento 1.5.

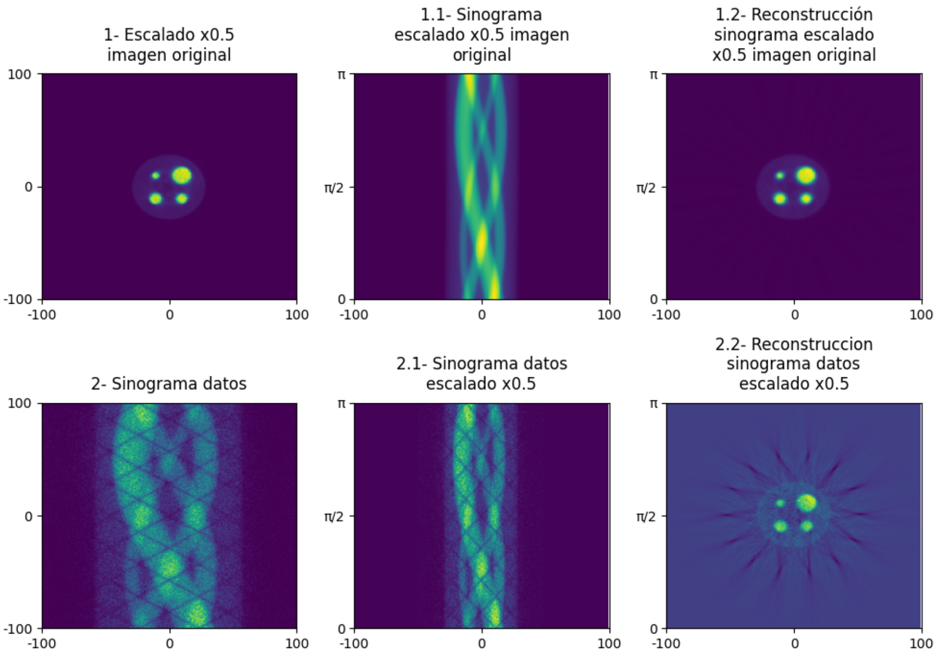

Figura 1.2: Resultados de aplicar una escalado de x05

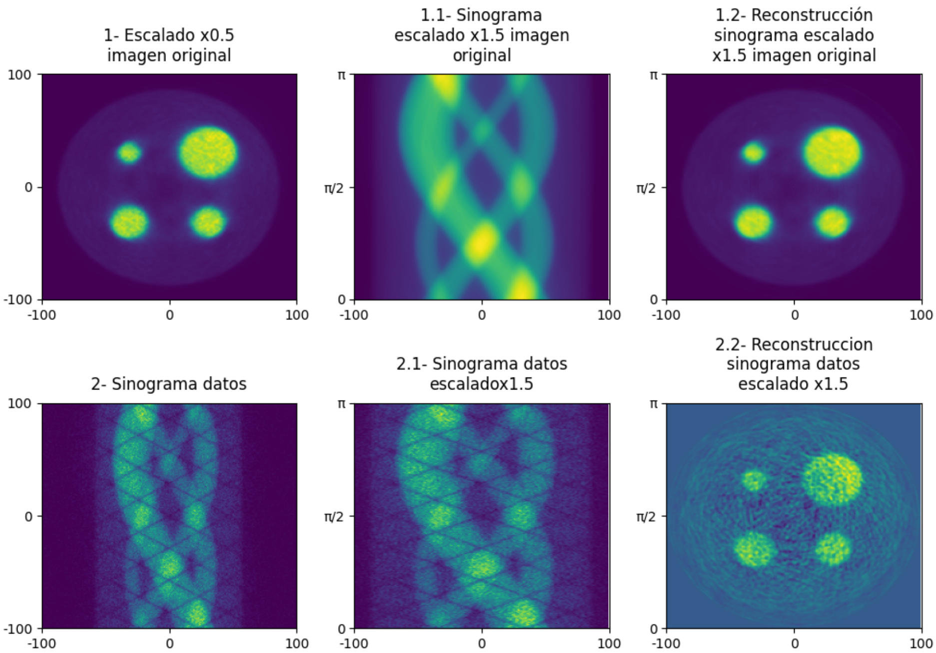

Figura 1.3: Resultados de aplicar una escalado de x1.5

4. Discusión

Vamos ahora a realizar una discusión cualitativa de los resultados obtenidos para ambos métodos desarrollados en este estudio.

4.1. Rotaciones

Si observamos las diferentes imágenes de la figura 1.1 . Por un lado las imágenes 1 y 2 muestran las imágenes de las que partimos antes de aplicar la transformación de rotación en el espacio de sinograma, la imagen 3 muestra la imagen en el espacio real una vez aplicada la transformación de rotación. Una vez aplicada las transformaciones haciendo uso de nuestros algoritmos en las imágenes 1.1 y 2.1 podemos observar que se han obtenido el mismo resultado que en la imagen 3.1, que es la considerada como la imagen verdadera a la hora de realizar las comprobaciones. Por lo tanto en todos estos casos se han obtenido la misma imagen. Con el fin de corroborar el correcto funcionamiento, también se ha realizado la reconstrucción de los sinogramas y en las figuras 1.2,2.2 y 3.2 podemos ver que al reconstruir las imágenes, realizando la transformada inversa de Radon, obtenemos de imagen real el mismo sistema. Por lo tanto, a la vista de los resultados, el método desarrollado funciona adecuadamente y es compatible con los resultados obtenidos si aplicamos la transformación directamente sobre la imagen real.

4.2. Escalado

Escalado x0.5

Observando ahora la figura 1.2 , podemos extraer las imágenes 1.1 y 2.1 nos devuelven los mismos sinogramas una vez se ha aplicado la transformacional. Así mismo en las imágenes 2.1 y 2.2, se realiza la transformada inversa de Radon y se obtiene la misma imagen en el dominio de la imagen real.

Escalado x1.5

Observando ahora la figura 1.3 , podemos inferir que las imágenes 1.1 y 2.1 nos devuelven los mismos sinogramas una vez se ha aplicado la transformación de escalado equivalente a un acercamiento de la imagen. Así mismo en las imágenes 2.1 y 2.2, se realiza la transformada inversa de Radon y se obtiene la misma imagen en el dominio de la imagen real.

Por lo tanto a la vista de los resultados que se han obtenido, el algoritmo desarrollado funciona correctamente a la hora de aplicar el escalado directamente sobre las imágenes el dominio de sinograma y es equivalente al aplicar el escalado a la imagen real y luego posteriormente convertirla al espacio sinograma.

5. Conclusiones

En este articulo se han introducido la base algorítmica para la aplicación de transformaciones de aumentado de datos directamente sobre imágenes el domino de los sinogramas. Se ha propuesto algoritmos para la aplicación de rotaciones y escalados, comprobando de forma experimental sobre un conjunto de datos simulado, que dichos métodos son equivalentes a la aplicación de las transformaciones directamente sobre las imágenes en el dominio real y su posterior conversión al espacio sinograma

6. Trabajo futuro

Como trabajo futuro, se podría extender los algoritmos de transformación sobre el espacio sinograma, implementando un algoritmo para la transformación de traslaciones o extender estas técnicas a otro tipo de imagen médica que también implique la transformación a otro tipo de dominio diferente al de la imagen real que se adquiere en la toma de datos.

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Introduction to Knot Theory

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1 Introduction

Knot theory is a branch of topology that has a rich history dating back to the 19th century and has profound implications in various fields such as physics, biology, and computer science. This article describes the main techniques to study this mathematical object, such as combinatorial, geometric and algebraic techniques, as well as some of the applications of this theory.

First of all, we define the concept of knot with a formal mathematical definition, which allows us to abstract the idea of knot. As a consequence of this, we define other important concepts in knot theory such as projections or diagrams of a knot.

In the following chapter, we introduce the Reidemeister Moves. With them, we use some combinatorial tools that helps us to classify different knots, specifically, the concepts of colorability, labelings and the Alexander polynomial.

In the next chapter, we study the relation between knot theory and geometry. Specifically, between surfaces and how knots can be a part of their boundary. First, we define the concept of surface and some important properties. Then, we introduce some classification theorems and we finish with the statement that for every knot we have a surface with the knot as its boundary.

After that, we will study some algebraic techniques. We begin with an introduction to group labelings, a similar concept to the p-labelings introduced earlier, how they interact with conjugacy classes and finally we will study homotopy in knots, namely the fundamental group and its connection to labelings.

Finally, in the last chapter, we study the applications of knot theory to different fields, such as physics or biology.

1.1 Definition of a knot

In this chapter, we first define the concept of knot with a formal mathematical definition, that allows us to abstract the idea of knot. We make this definition with polygonal curves to extend it later to general curves. With the definition of a knot we introduce too the concepts of oriented knot and link.

After that, we have to determinate when two knots are the same knot. We introduce the concept of elementary transformation and when two knots are equivalent.

By definition, the knots are subspaces of $\mathbb{R}^3$, so to represent them in the plane, we can use projections. The problem is that in some cases we can lose information about the knot in the intersections if we only have the projections. To avoid that, we introduce the definition of diagram of a knot.

2 Main results

2.1 Combinatorial techniques

In the following chapter, we introduce the Reidemeister Moves. With them, we use some combinatorial tools that helps us to classify the different knots. Specifically, the concepts of colorability, labelings and the Alexander polynomial.

First of all, we introduce six elementary deformations, the Reidemeister Moves. The importance of this tool is that every pair of equivalents knots differ in a finite number of this special deformations.

The next part is the colorability, a method of distinguishing knots by coloring their diagrams. We prove that if a knot is colorable, every equivalent knot is colorable too, because the colorability is not affected by the Reidemeister Moves. This is an important result because if we have two knots where one of them is colorable and the other is not, we can affirm they are not equivalent.

The concept of colorability can be generalized to p colors, called p-labeling, where p is a prime number. It has the same property of being invariant under Reidemeister Moves, so with it we can classify a lot of new knots.

On the last part of the section, we introduce the Alexander polynomial. It consists of associating a polynomial to each oriented knot. We also include a theorem that states that if we have two different diagrams of a knot, these differ by the multiplication of a monomial of the form $\pm t^k$, where k is an integer.

2.2 Geometric techniques

In this chapter we study the relation between knot theory and geometry. Specifically, between surfaces and how knots can be a part of their boundary. First, we define the concept of surface and some important properties. Then, we introduce some classification theorems and we finish with the statement that for every knot we have a surface with the knot for its boundary.

Definition 1 *Given three non-collinear points in 3-space, p_1, p_2 and p_3, we can define a triangle by the set of points*

$$\{xp_1 + yp_2 + zp_3 \mid x + y + z = 1,\ x, y, z \geq 0\}, \tag{1.1}$$

where each p_i is thought of as a vector in $\mathbb{R}^3$. Now, let's define polyhedral surface.

Given a triangle as described in Eq. (1.1), we define polyhedral surface as the union of a finite collection of triangles that satisfy

1. *each pair of triangles is either disjoint or their intersection is a common edge or vertex*
2. *at most two triangles share a common edge*
3. *the union of the edges that are contained in exactly one triangle is a disjoint collection of simple polygonal curves, called the boundary of the surface.*

With this definition we introduce the concept of polyhedral surface. We can introduce two important properties.

Definition 2 *A polyhedral surface is orientable if it is possible to orient the boundary of each of its constituent triangles in such a way that when two triangles meet along an edge, the two induced orientations of that edge run in opposite directions. This property is independent of the choice of triangulation.*

Definition 3 *Polyhedral surfaces are called homeomorphic if, after some subdivision of the triangulations of each, there is a bijection between their vertices such that when three vertices in one surface bound a triangle the corresponding three vertices in the second surface also bound a triangle.*

Thanks to this last definition, we can classify our surfaces, looking if they are homeomorphic or not. In this matter, we have the classification theorems. These theorems establish if two surfaces are homeomorphic by looking at their properties.

Another important concept of oriented surfaces is the genus.

Definition 4 *The genus of a connected orientable surface S is given by*

$$g(S) = \frac{2 - \chi(S) - B}{2}, \tag{1.2}$$

where B is the number of boundary components of the surface.

We finish this section with the theorem that establishes the relation between knots and surfaces.

Theorem 1 *Every knot is the boundary of an orientable surface.*

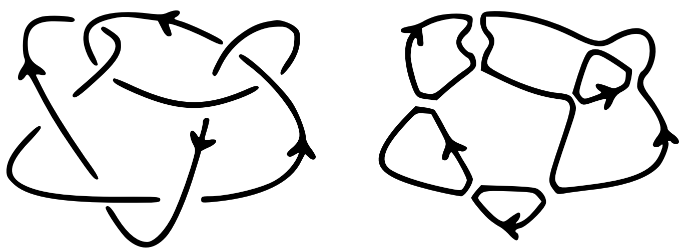

Figure 1.1: First part of the proof.

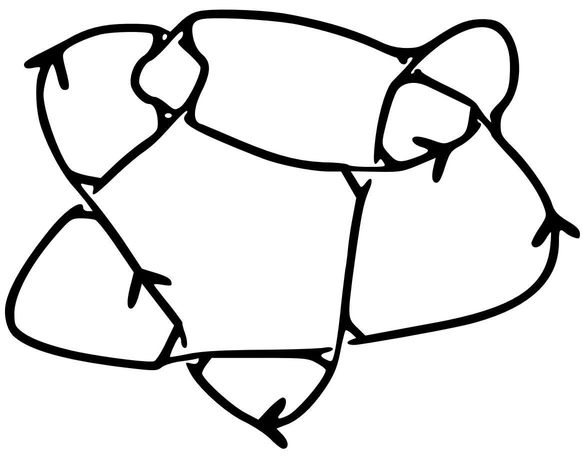

Figure 1.2: Second part of the proof.

2.3 Algebraic techniques

In this chapter, we study group labelings, a similar concept to the p-labelings introduced in 2.1, as well as homotopy in groups, namely the fundamental group, and its connection to labelings.

Given an oriented knot diagram and a group G (we focus on symmetric groups), a labeling of the knot with elements of the group is defined by assigning an element of G to each arc of the knot diagram, satisfying two requirements:

1. *Consistency*: at any given crossing, if the arcs are labeled $g, h, k \in G$ according to Fig. 1.3, then they must fulfill $gkg^{-1} = h$ or $ghg^{-1} = k$, if the crossing is right-handed or left-handed, respectively.
2. *Generation*: the set of all labels must generate the group G.

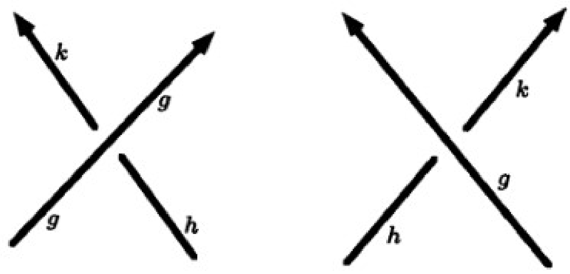

Figure 1.3: Consistency rule in group labelings. Right and left-handed crossings, respectively.

These labelings help us distinguish different knots, thanks to the following theorem:

Theorem 2 *If a knot diagram can be labeled with elements from a group G, then any diagram of the same knot can also be labeled with elements from the group, regardless of the orientation.*

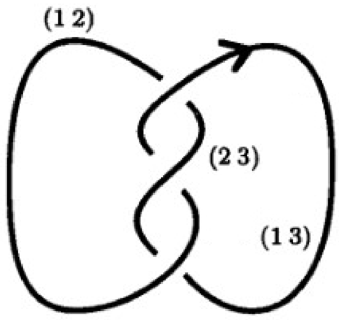

Figure 1.4: S_3-labeling of the trefoil knot.

Given a knot diagram labeled with elements from a group, it is easy to see that all the labels in the diagram are part of the same conjugacy class, due to the consistency condition. This lets us extend Theorem 2 to a stronger version, except orientation can't be ignored anymore, since it is possible for an element of a group to not be conjugate to its inverse.

Theorem 3 *If a knot diagram can be labeled with elements from a group G, where the labels come from a certain conjugacy class of the group, then any diagram of the same knot can also be labeled with elements from the same conjugacy class.*

Due to the consistency condition, we only need some labels in order to completely determinate the entire labeling, therefore to find labelings we must simply find those initial labels and solve the equations that arise from the consistency at each crossing, and as a result of the generation condition, the initial labels must also generate the group, since the induced labels are already generated by those first labels.

Using these initial labels and equations, we can construct a group by presentation, where the labels are the generators and the equations are the relations or words. We denote this by $G(K) = \langle x_n \mid w_m = 1 \rangle$, where x_n are the generators, $w_m = 1$ are the relations and K is the knot. This group is called the knot group. It can be shown that groups determined by different diagrams of the same knot are isomorphic.

Given a knot K as an embedding in $\mathbb{R}^3$, we can consider the fundamental group of the space $\mathbb{R}^3 \setminus K$, denoted π_1. This lets us study the knot without depending on a diagram of the knot. In fact, this group is actually isomorphic to the knot group we defined previously.

Theorem 4 *If K is a knot, then $G(K)$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus K)$.*

3 Applications of knot theory

In this chapter we study the applications of knot theory to different fields, such as physics or biology, following Chapters 12 and 13 of [2].

The first mention of knot theory was done by William Thomson, in his atom theory, in which was stated that the chemical properties of elements were related to the knots between the atoms. The first paper of knot theory was first published by Peter Tait.

We can mention two applications of knot theory in other fields. One in physics and the other in biology. In the first area, our theory is used in the research field of Statistical Mechanics. They have created a statistical mechanical model that copies the matter in an ideal way with the aim of studying it better. In this model, there's a function, called the partition function, related to the invariants of knots.

In biology, the application is seen in the study of the DNA. It was discovered that the information contained in the DNA molecules is independent of the knots between them, but they influence their function in the cell.

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Lie Algebras, Root Systems and Classification

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1 Basic definitions and concepts

A Lie algebra over a field F is an F-vector space L, together with the Lie bracket, a bilinear map:

$$L \times L \to L,\ (x, y) \mapsto [x, y],$$

that verify the properties:

- $[x, x] = 0,\ \forall x \in L,$
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,\ \forall x, y, z \in L.$

Note that any vector space V has a Lie bracket defined by $[x, y] = 0,\ \forall x, y \in V$ and this is the abelian Lie algebra structure on V. With these definitions, we are able to introduce the concept of subalgebra, ideal and homomorphism. Let L be a Lie algebra, a vector subspace $K \subseteq L$ such that $[x, y] \in K,\ \forall x, y \in K$ is a Lie subalgebra of L and a vector subspace $I \subseteq L$ such that $[x, y] \in I,\ \forall x \in L, y \in I$ is an ideal. Moreover, to "connect" two different Lie algebras over the same field, we have to talk about homomorphisms, mappings that preserves the Lie bracket. In particular, there is an homomorphism we will use frequently: the adjoint homomorphism. This is defined as $\text{ad} : L \to \mathfrak{gl}(L)$, where $(\text{ad}\ x)(y) := [x, y],\ \forall x, y \in L$, where $\mathfrak{gl}(L)$ is the Lie algebra of the vectorial endomorphisms of L with the Lie bracket $[x, y] = x \circ y - y \circ x$.

The derived algebras are important in the classification of the Lie algebras as well as the centre of a Lie algebra is. The derived algebra of a Lie algebra L is $L' := [L, L] = \text{Span}\{[x, y] : x, y \in L\}$. In fact, the criteria of classification of Lie algebras of dimensions 2 and 3 are based on the properties of L', its derived algebra, and $Z(L)$, its centre. This Lie algebras are important because they appears often as subalgebras of bigger Lie algebras.

We introduce the representation theory, where the main objective is to explore the methods through which an abstract Lie algebra can be concretely interpreted as a subalgebra within the endomorphism algebra of a finite-dimensional vector space. So a representation is an homomorphism from a Lie algebra to $\mathfrak{gl}(V)$ for some vector space V. Futhermore, we introduce modules and submodules, note that the concept of a module over a Lie algebra is a generalization of the notion of a linear representation of a group. By relating the concepts of module and submodule, the Lie module V is said to be irreducible, or simple, if it is non-zero and it has no submodules other than $\{0\}$ and V.

2 Finite-dimensional irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$

Consider the vector space $\mathbb{C}[X,Y]$. For each integer $d \geq 0$, we define V_d as the subspace composed of homogeneous polynomials in X and Y of degree d. In this context, V_0 represents the 1-dimensional vector space of constant polynomials. For any ≥ 1, the space V_d is based on the monomials $X^d, X^{d-1}Y, ..., XY^{d-1}, Y^d$. This basis indicates that the dimension of V_d as a $\mathbb{C}$-vector space is $d+1$. We transform V_d into an $\mathfrak{sl}(2,\mathbb{C})$-module by defining a Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V_d)$. Given that $\mathfrak{sl}(2,\mathbb{C})$ is linearly spanned by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad ; \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad ; \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the mapping ϕ can be determined once we have defined $\phi(e), \phi(f)$ and $\phi(h)$. We have to note that the $\mathfrak{sl}(2,\mathbb{C})$-module V_d is irreducible.

It's evident that the $\mathfrak{sl}(2,\mathbb{C})$-modules V_d can't be isomorphic for different d because they have different dimensions. Furthermore, any finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-module can be shown to be isomorphic to a particular V_d. This can be proven examining the eigenvectors and eigenvalues of h.

Sometimes, a module for a Lie algebra may not be fully reducible, witch means that cannot be expressed as a direct sum of irreducible submodules. However, finite-dimensional representations of complex semisimple Lie algebras exhibit a much better behaviour, as we can verify, with the help of the Weyl's theorem, that if L be a complex semisimple Lie algebra then every finite-dimensional representation of L is completely reducible.

3 Root space decomposition

Given a semisimple complex Lie algebra L, we may consider H to be maximal subalgebra of L regarding that H is abelian with each element semisimple. Using this algebra, we can recover information about L. Defining

$$L_\alpha := \{x \in L : [h,x] = \alpha(h)x \text{ for all } h \in H\}$$

for each $\alpha \in H^*$, we can decompose L as

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

for the set Φ of nonzero $\alpha \in L^*$ with nonzero L_α. These α are called roots of L. Indeed, we can associate to each root α a Lie subalgebra of L isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. The presence of H in the previous decomposition of sum of weight spaces is not trivial, and it requires some work to do. It has to be proven that, as a matter of fact, L always contains a nonzero Cartan subalgebra H and that $L_0 = C_L(H) = H$.

4 Root systems

Let E be a finite-dimensional $\mathbb{R}$-vector space endowed with an inner product $(\cdot,\cdot)$. Then $R \subseteq E$ is a root system if the following properties hold:

(R1) R is finite, $\mathrm{Span}_{\mathbb{R}}(R) = E$ and $0 \notin R$,

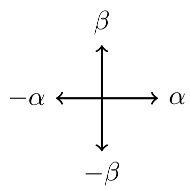
(R2) The only scalar multiples of $\alpha \in R$ are $\pm\alpha$, (R3) If $\alpha \in R$ then s_α is a permutation of R,

(R4) $\langle \beta, \alpha \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R$,

where $\langle x, v \rangle := \frac{2(x,v)}{(v,v)}$ and $s_v(x) = x - \frac{2(v,x)}{(v,v)}v$ fore ach $x \in E$.

The elements of R are the roots. Moreover, it holds that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$. This statement allows us to subsequently define the Dynkin diagrams. A very important property that Root Systems may have is the irreducibility. A root system R is irreducible if R cannot be expressed as a disjoint union of two nonempty subsets $R_1 \cup R_2$ such that $(\alpha, \beta) = 0$ for $\alpha \in R_1$, $\beta \in R_2$. The irreducibility of a root system shall tell us about the simplicity of a complex semisimple Lie algebra, and about the connectivity of what we will see below which are the Dynkin diagrams. Another important term when talking about root systems is a basis for a root system R of E. These are $\mathbb{R}$-basis of E such that each element of R can be expressed as a $\mathbb{Z}$-linear combination of the elements of the basis, where all the coefficients have the same sign. Certainly, each root system has a basis.

A more visual example of a root system is



,

where $\alpha = (1, 0)$ and $\beta = (0, 1)$.

An example of a root system is the set of roots of a complex semisimple Lie algebra L with respect to a fixed Cartan subalgebra H, say Φ. The proof of this is not trivial and requires some work. In particular, our associated Lie subalgebras to a root α isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and the properties of the Killing form of L are highly used, where the Killing Form of L is the symmetric bilinear form defined as

$$\kappa(x, y) := \text{tr } (\text{ad } x \circ \text{ad } y), \text{ for all } x, y \in L.$$

In fact, the Killing Form is a powerful tool when studying general properties of complex semisimple Lie algebras, as it allows us to characterize whether these are solvable or semisimple. These criterions in terms of the Killing Form are popularly known as Cartan's Criterions. Furthermore, this criterions can be used to prove that a semisimple Lie algebra is the direct product of simple Lie algebras.

5 Dynkin diagrams

Given a root system R and a basis B, the Coxeter graph of R is defined as the graph which vertices are the elements of B and the edges connecting two vertices is $d_{\alpha,\beta} = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$. The Dynkin diagram of R with basis B is the Coxeter graph adding arrows to the edges following the following rule: if $d_{\alpha,\beta} > 1$ then we add an arrow to the edge connecting α and β pointing from the longer vector to the shorter vector. Is important to notice that when $d_{\alpha,\beta} > 1$ there are always a strictly larger vector so it is well defined. Other important fact is that the Dynkin diagram of a root system does not depend on the chosen basis, so we can speak about the Dynkin diagram of a root system.

With that we have all the needed tools to start the classification of complex semisimple Lie algebras.

6 Classification of Complex Semisimple Lie Algebras

Why we want to classify the complex semisimple Lie algebras? Besides from the role that the complex simple Lie algebras have in other branches of mathematics, they have a very important role in the Lie theory itself. When doing an induction proof it is not uncommon to end up with the simple case, so having controlled the simple Lie algebras help us to finish our proof.

Our goal is to find a one-to-one correspondence between complex semisimple Lie algebras (up to isomorphism) and Dynkin diagrams, and classify the Dynkin diagrams.

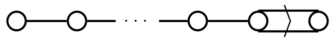
Given a complex semisimple Lie algebra and a Cartan subalgebra, we can construct a root system, so we have a Dynkin diagram. Is important to mention that this Dynkin diagram does not depend on the Cartan subalgebra, so we can talk about the Dynkin diagram of the Lie algebra. Furthermore, it is invariant up to isomorphism. So we have a mapping between the isomorphism classes of complex semisimple Lie algebras and the Dynkin diagrams.

Since semisimple Lie algebras are direct sum of simple Lie algebras, we can focus on the study of complex simple Lie algebras. The Dynkin diagrams that arise from complex simple Lie algebras are exactly the connected ones, moreover, the reciprocal is also true, so if a complex semisimple Lie algebra has a connected Dynkin diagram then it is simple. The relation of the Dynkin diagram of a complex semisimple Lie algebra and the Dynkin diagrams of is simple ideals is exactly what we can expect, the Dynkin diagrams of the simple ideals are the connected components of the Dynkin diagram of the semisimple Lie algebra.

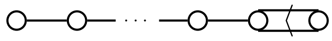
The classification of the connected Dynkin diagrams is as follows:

- A_l for $l \geq 1$:

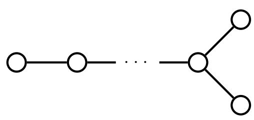
- B_l for $l \geq 2$:



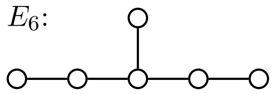
- C_l for $l \geq 3$:



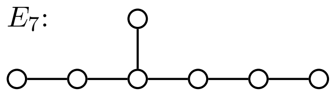
- D_l for $l \geq 4$:



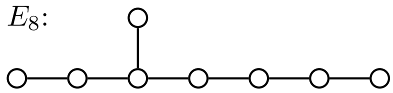
- E_6:

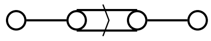


- E_7:



- E_8:



- F_4: 

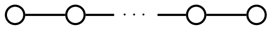
- G_2: 

Where the diagrams A_l, B_l, C_l and D_l have l nodes. We have four infinite families, and five exceptional cases.

Now that we connected Dynkin diagrams, we can start studying the correspondence in order to see how much of the classification we extrapolate to the complex semisimple Lie algebras. So we want to know if the correspondence is one-to-one. The answer is yes, if two complex semisimple Lie algebras have the same Dynkin diagrams they are isomorphic. And given a Dynkin diagram we can construct a complex semisimple Lie algebra such that its Dynkin diagram is the given diagram, this is the Serre's theorem.

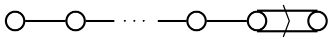
The problem of Serre's theorem is that the constructed Lie algebra does not have a practical presentation. So we present a classical examples of complex semisimple Lie algebras that are easy to work with. The classical Lie algebras, $\mathfrak{sl}(2l+1), \mathfrak{so}(2l+1,\mathbb{C}), \mathfrak{so}(2l,\mathbb{C})$ and $\mathfrak{sp}(2l,\mathbb{C})$ where $l \geq 1$, are subalgebras of the complex general linear Lie algebra, and have a simple basis. Except $\mathfrak{so}(2,\mathbb{C})$, which is abelian, they are semisimple, and have the following Dynkin diagrams:

- $\mathfrak{sl}(l+1, \mathbb{C}),\ l \geq 1$:



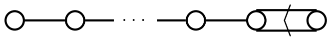
- $\mathfrak{so}(2l+1, \mathbb{C}),\ l = 1$:

- $\mathfrak{so}(2l+1, \mathbb{C}),\ l \geq 2$:



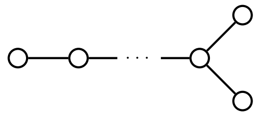
- $\mathfrak{sp}(2l, \mathbb{C}),\ l = 1$:

- $\mathfrak{sp}(2l, \mathbb{C}),\ l \geq 2$:



- $\mathfrak{so}(2l, \mathbb{C}),\ l = 2$:

- $\mathfrak{so}(2l, \mathbb{C}),\ l \geq 3$:



where every diagram have l nodes.

Using the previous results, we can immediately see that the classical Lie algebras are simple except $\mathfrak{so}(2, \mathbb{C})$ and $\mathfrak{so}(4, \mathbb{C})$. The families A_l, B_l, C_l and D_l correspond to classical Lie algebras. And we can deduce some isomorphism that are difficult to come with directly, for example $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$ or $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

In conclusion:

- We have seen that the Dynkin diagram determine unequivocally the complex semisimple Lie algebra up to isomorphism, so the classification of the Dynkin diagrams can be extrapolated.
- The complex simple Lie algebras are one of the families A_l $(l \geq 1)$, B_l $(l \geq 2)$, C_l $(l \geq 3)$ or D_l $(l \geq 4)$, which correspond to classical Lie algebras, or is one of the five exceptional cases E_6, E_7, E_8, F_4 or G_2. We are only left with five simple Lie algebras that we do not know a friendly presentation.
- The complex semisimple Lie algebras are classified as well, since they are direct sum of complex simple Lie algebras.

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Composition operators on Gelfand-Shilov classes.

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1 Introduction

Given a function $\psi : \mathbb{K}^N \to \mathbb{K}^N$ and a suitable family of functions X defined on $\mathbb{K}^N$, the composition operator associated with ψ on X is $C_\psi f = f \circ \psi$, for every $f \in X$. Given a topological vector space X, a relevant and not always obvious problem is to find necessary and sufficient conditions on ψ for $C_\psi(X) \subset X$ and $C_\psi : X \to X$ to be continuous.

The composition operators and their properties have been studied in several topological vector spaces such as in the space of holomorphic functions, in the space of real analytic functions (see for instance [6, 15, 16] and the references therein), in the space of smooth functions (see for instance, [14] and the references therein) and also in the Schwartz space (see for instance [8, 9, 10]). There are many classical problems related to this operator (see for instance [4] and the references therein).

It is well-known the following classical result:

Theorem 1.1 (Borel's theorem) *Any formal series $\sum_{j=0}^{\infty} c_j x^j$ is the Taylor series of a smooth function defined in an open neighborhood of the origin. In other words, the Borel map $B : C^\infty(\mathbb{R}) \to \mathbb{R}^{\mathbb{N}}$ defined by $B(f) = (f^{(j)}(0))_j$ is surjective.*

From this, we see at once that the space of smooth functions is much "bigger" than the space of real analytic functions. It would be interesting to find intermediate families of functions to parametrize the gap existing between both. Are there spaces between one and the other that have "nice" properties and for which the composition operator is worth studying? It turns out that there is a family of classes that gives an affirmative answer to the previous question:

Definition 1.1 *The Gevrey class (of index $s \geq 1$) $G^s(\mathbb{R})$ is defined as the set of smooth functions f such that for every compact subset K there exists a $C = C_{K,f} > 0$ satisfying that*

$$\sup_{x \in K} |f^{(j)}(x)| \leq C^{j+1} (j!)^s$$

for all $j \in \mathbb{N}_0$.

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On the one hand, if f is a real analytic function it is easy to see using Cauchy's integral formula that for every compact subset K there exists a $C = C_{K,f} > 0$ satisfying that

$$\sup_{x\in K} |f^{(j)}(x)| \leq C^{j+1}\, j!$$

for all $j \in \mathbb{N}_0$. So $f \in G^1(\mathbb{R})$. On the other hand, if $f \in G^1(\mathbb{R})$ then, it holds by Cauchy–Hadamard theorem that f is real analytic. So $G^1(\mathbb{R}) = \mathcal{A}(\mathbb{R})$. It's trivial that $G^s(\mathbb{R}) \subset G^{s+h}(\mathbb{R}) \subset C^\infty(\mathbb{R})$, for all $s \geq 1, h > 0$. Furthermore, the following facts are well-known:

- For $s > 1$, $G^s(\mathbb{R})$ is an algebra.
- For $s > 1$, $G^s(\mathbb{R})$ is both closed under differentiation and under composition.
- The Inverse Function theorem holds in $G^s(\mathbb{R})$ for every $s \geq 1$.

These classes appeared for the first time in the work of Gevrey, who measured the growth behaviour of such functions in terms of a weight sequence $(M_p)_p$, which is $((p!)^s)_p$, $s \geq 1$, in the Gevrey case and which satisfies certain technical conditions in the general case of $(M_p)-$ultradifferentiable classes. Later Beurling [2] pointed out that one can also use weight functions ω to measure the smoothness of functions with compact support by the decay properties of their Fourier transform. This method was modified by Braun, Meise, and Taylor in [5], who showed that also these classes can be defined by the decay behaviour of their derivatives, if one uses the Young conjugate of the function $t \to \omega(e^t)$. Meise and Taylor in [12] showed that under rather strong conditions both ways lead to the same class. But in general there are classes defined in one way which cannot be defined in the other way. See [3] for more details. The composition operator on the case of $\omega-$ultradifferentiable functions has been studied in [7].

Recall that the Schwartz class $\mathcal{S}(\mathbb{R})$ consists of those smooth functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$p_n(f) := \sup_{x\in\mathbb{R}} \sup_{1\leq j\leq n} (1+x^2)^n |f^{(j)}(x)| < \infty$$

for each $n \in \mathbb{R}$. It turns out that $\mathcal{S}(\mathbb{R})$ is a Fréchet space when it is endowed with the topology generated by the sequence of seminorms $(p_n)_{n\in\mathbb{N}}$.

The Gevrey classes are made of functions whose derivatives verify certain local estimations, whereas the Schwartz class is made of functions whose derivatives asymptotically decrease fast "enough". Combining both the Gevrey classes and the Schwartz class, we define the following well-known family of smooth functions (originally introduced in [11], see [13] and the references therein for further information):

Definition 1.2 *The Gelfand-Shilov space $\Sigma_d(\mathbb{R})$, with $d > 1$, consists of those functions $f \in C^\infty(\mathbb{R})$ such that, for each $h > 0$:*

$$\sup_{x\in\mathbb{R}} \sup_{j,\ell\in\mathbb{N}_0} \frac{|x^\ell f^{(j)}(x)|}{h^{j+\ell}[(j+\ell)!]^d} < +\infty.$$

We can define more general families of functions, by changing the sequence $([(j+\ell)!]^d)_{j,\ell}$ above for a suitable sequence $(M_{j+\ell})_{j,\ell}$, called weight sequence.

Definition 1.3 *A sequence $(M_p)_{p\in\mathbb{N}_0}$ is a weight sequence if it satisfies*

(M0) There exists $c > 0$ such that $\big(c(p+1)\big)^p \leq M_p,\ p \in \mathbb{N}_0$.

(M1) $M_p^2 \leq M_{p-1}M_{p+1},\ p \in \mathbb{N}$ *and* $M_0 = 1$.

(M2) There are $A, H > 0$ *such that* $M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q},\ p \in \mathbb{N}_0$.

(γ_1) $\sup_p \frac{m_p}{p} \sum_{j\geq p} \frac{1}{m_j} < \infty$, *where* $m_p = \frac{M_p}{M_{p-1}}$.

Definition 1.4 *The space* $\mathcal{S}_{(M_p)}(\mathbb{R})$ *associated to the weight sequence* $(M_p)_{p\in\mathbb{N}_0}$ *consists of those functions* $f \in C^\infty(\mathbb{R})$ *such that, for each* $h > 0$:

$$\sup_{x\in\mathbb{R}} \sup_{j,\ell\in\mathbb{N}_0} \frac{|x^\ell f^{(j)}(x)|}{h^{j+\ell} M_{j+\ell}} < +\infty.$$

As we have hinted above, we can define the following global class of smooth functions using weight functions instead of weight sequences in the following way:

Definition 1.5 *A continuous increasing function* $\omega : [0, \infty[\longrightarrow [0, \infty[$ *is called a weight if it satisfies:*

(α) *there exists* $K \geq 1$ *with* $\omega(2t) \leq K(\omega(t) + 1)$ *for all* $t \geq 0$,

(β) $\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty$,

(γ) $\log(1 + t^2) = o(\omega(t))$ *as* t *tends to* ∞,

(δ) $\varphi_\omega : t \to \omega(e^t)$ *is convex.*

The function ω is extended to $\mathbb{R}$ as $\omega(x) = \omega(|x|)$. The *Young conjugate* $\varphi^*_\omega : [0, \infty[\longrightarrow \mathbb{R}$ of φ_ω is defined by

$$\varphi^*_\omega(s) := \sup\{st - \varphi_\omega(t) :\ t \geq 0\},\ s \geq 0.$$

Then φ^*_ω is convex, $\varphi^*_\omega(s)/s$ is increasing and $\lim_{s\to\infty} \frac{\varphi^*_\omega(s)}{s} = +\infty$. Moreover, for every $A > 0$, $\lambda > 0$ there is $C > 0$ such that

$$A^j j! \leq Ce^{\lambda\varphi^*_\omega(\frac{j}{\lambda})}$$

for each $j \in \mathbb{N}_0$. The weight function ω is said to be a *strong weight* if

(ε) there exists a constant $C \geq 1$ such that for all $y > 0$ the following inequality holds

$$\int_1^\infty \frac{\omega(yt)}{t^2}\, dt \leq C\omega(y) + C. \tag{1.1}$$

Definition 1.6 *Let* ω *be a weight function. The Gelfand-Shilov space of Beurling type* $\mathcal{S}_{(\omega)}(\mathbb{R})$ *consists of those functions* $f \in L^1(\mathbb{R})$ *with the property that* $f, \widehat{f} \in C^\infty(\mathbb{R})$ *and*

$$q_{\lambda,j}(f) := \max\big(\sup_{x\in\mathbb{R}} |f^{(j)}(x)|e^{\lambda\omega(x)},\ \sup_{\xi\in\mathbb{R}} |\widehat{f}^{(j)}(\xi)|e^{\lambda\omega(\xi)}\big) < +\infty$$

for every $\lambda > 0, j \in \mathbb{N}_0$.

$\mathcal{S}_{(\omega)}(\mathbb{R})$ is a Fréchet space with different equivalent systems of seminorms. In particular we shall use the families of seminorms

$$p_\lambda(f) := \sup_{j,k\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} |x^k f^{(j)}(x)| e^{-\lambda\varphi^*_\omega(\frac{j+k}{\lambda})}, \quad \lambda > 0$$

or

$$\pi_{\lambda,\mu}(f) := \sup_{j\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} |f^{(j)}(x)| e^{-\lambda\varphi^*_\omega(\frac{j}{\lambda})+\mu\omega(x)}, \quad \lambda > 0, \mu > 0.$$

Let $d > 1$ be given. The Gelfand-Shilov space $\Sigma_d(\mathbb{R})$ is

$$\Sigma_d(\mathbb{R}) = \mathcal{S}_{(M_p)}(\mathbb{R}) = \mathcal{S}_{(\omega)}(\mathbb{R}),$$

where

$$M_p = p!^d, \quad \omega(t) = t^{\frac{1}{d}}.$$

The two approaches described are nonequivalent, it is strictly more general the one that uses weight functions. This is one of the reasons we favour working with weight functions ω instead of weight sequences $(M_p)_p$. In [1] we started the investigation of composition operators on the Gelfand-Shilov space of Beurling type $\mathcal{S}_{(\omega)}(\mathbb{R})$. The following three facts about $\mathcal{S}_{(\omega)}(\mathbb{R})$ play an important role in the proof of the results below:

- $\mathcal{S}_{(\omega)}(\mathbb{R})$ is Montel (i.e. bounded and closed sets are compact).
- Condition (β) is equivalent to the existence of non-trivial functions with compact support on $\mathcal{S}_{(\omega)}(\mathbb{R})$.
- The following version of Borel's theorem holds in our setting when condition (ε) holds for ω: the Borel map
$$B : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\{0\}), f \mapsto \left(f^{(j)}(0)\right)_{j\in\mathbb{N}_0},$$
where
$$\mathcal{E}_{(\omega)}(\{0\}) = \left\{(x_j)_j \in \mathbb{C}^{\mathbb{N}_0} : \sup_j |x_j| \exp(-k\varphi^*_\omega(\frac{j}{k})) < \infty \ \forall k > 0\right\}.$$

2 Main results

It is obvious that if ψ is a polynomial then $f \circ \psi$ belongs to the Schwartz class whenever f does. We find that this is no longer true in $\Sigma_d(\mathbb{R})$. More generally, we have the following result:

Theorem 2.1 *Let $d > 1$ and $\psi \in C^\infty(\mathbb{R})$ be given such that $C_\psi(\Sigma_d(\mathbb{R})) \subset \Sigma_d(\mathbb{R})$. Then ψ' is bounded.*

It remains unknown whether the condition $C_\psi(\Sigma_d(\mathbb{R})) \subset \Sigma_d(\mathbb{R})$ implies that $\psi^{(\ell)}$ is bounded for all $\ell > 1$. Since $\Sigma_d(\mathbb{R}) \subset \Sigma_{d'}(\mathbb{R})$ whenever $1 < d < d'$, we can investigate the optimal index d' for which $C_\psi(\Sigma_d(\mathbb{R})) \subset \Sigma_{d'}(\mathbb{R})$ holds for any non-constant polynomial ψ. We have the following result:

Theorem 2.2 *Let ψ be a function satisfying $\lim_{x\to+\infty} |\psi(x)| = +\infty$ and $|\psi'(x)| \geq c|\psi(x)|^k$ for some $c > 0$ and $x > 0$ large enough. Then for every $d \leq d' < (k+1)d$ there exists $f \in \Sigma_d(\mathbb{R})$ such that $f \circ \psi \notin \Sigma_{d'}(\mathbb{R})$.*

Let ψ be a polynomial of degree $N > 1$. Putting $k = \frac{N-1}{N}$ in the previous result we obtain the following consequence:

Corollary 2.3 *Let ψ be a polynomial of degree $N > 1$. Then for every $d \leq d' < \frac{2N-1}{N} d$ there is $f \in \Sigma_d(\mathbb{R})$ such that $f \circ \psi \notin \Sigma_{d'}(\mathbb{R})$. In particular, for any polynomial ψ of degree greater than one and $d \leq d' < \frac{3}{2} d$ there is $f \in \Sigma_d(\mathbb{R})$ such that $f \circ \psi \notin \Sigma_{d'}(\mathbb{R})$.*

Is there any $d' > d$ so that $C_\psi\left(\Sigma_d(\mathbb{R})\right) \subset \Sigma_{d'}(\mathbb{R})$ for any polynomial ψ? The answer is affirmative:

Theorem 2.4 *Let $d > 1$. If ψ is a non constant polynomial then $f \circ \psi \in \Sigma_{2d}(\mathbb{R})$ for every $f \in \Sigma_d(\mathbb{R})$. In other words, $C_\psi : \Sigma_d(\mathbb{R}) \to \Sigma_{2d}(\mathbb{R})$.*

Since $C_\psi \circ C_\psi = C_{\psi \circ \psi}$, the previous result allows us to study the dynamical properties of the composition operator on Gelfand-Shilov classes in forthcoming works.

Now we turn our attention to other important family of weight functions. Consider the weight function $\omega(t) = \max\{0, \log^s(t)\}$, with $s > 1$. For $s = 1$, ω is not a weight function. It is an extreme case of weight function for which $\mathcal{S}_{(\omega)}(\mathbb{R}) = \mathcal{S}(\mathbb{R})$. In this family of weight functions we recover the result that we had for $\mathcal{S}(\mathbb{R})$:

Theorem 2.5 *Consider $\omega(t) = \max\{0, \log^s(t)\}$ with $s > 1$. For every polynomial ψ, it holds that $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$.*

In [1] we also study the compactness of the composition operator C_ψ on Gelfand-Shilov classes and obtain the following similar result to the one obtained in [10] but with a different approach:

Theorem 2.6 *Let ω be a strong weight and let us assume that $\psi \in C^\infty(\mathbb{R})$ satisfies the condition $C_\psi(\mathcal{S}_{(\omega)}(\mathbb{R})) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$. Then $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is not a compact operator.*

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