CONVERGENCE ANALYSIS OF B-SPLINE MULTIDIMENSIONAL DEFORMABLE MODELS DEFINED IN THE FREQUENCY DOMAIN

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ABSTRACT

This abstract presents a formulation of multidimensional deformable models from a d-dimensional generalization of Liang et al., using B-splines as shape function and a frequency-based formulation. The convergence of theses models is analyzed showing that it depends of the dynamic parameters of the model and the spectral characteristics of the data. This result allows us to apply the frequency formulation to multidimensional data, as an efficient iterative system in matrix form in several applications such as fast segmentation and motion tracking of d-dimensional objects with non-rigid motion or deformation.

A dynamic deformable model is defined as a parametric hypersurface in $\mathbb{R}^d$, $\mathbf{v}(\mathbf{s},t)$, where $\mathbf{s} \equiv [s_1,\ldots,s_e]$ with $e \leqslant d-1$, is the vector of the parametric variables of the space domain. The model is governed by an energy functional $\mathcal{E}(\mathbf{v}) = \mathcal{S}(\mathbf{v}) + \mathcal{P}(\mathbf{v})$. The first term is the internal deformation energy,

$$\mathcal{S}(\mathbf{v}) = \frac{1}{2}\sum_{l=1}^{d}\left(\int_\Omega \alpha(\mathbf{s})\left\|\nabla v_l(\mathbf{s})\right\|^2 + \beta(\mathbf{s})\left|\Delta v_l(\mathbf{s})\right|^2 d\mathbf{s}\right) \tag{1}$$

where α and β control the elasticity and the rigidity in any coordinate $\mathbf{s}$ of the model. The term $\mathcal{P}(\mathbf{v})$ contains the rest of energies applied to the model.

According to the variational calculus, the model $\mathbf{v}(\mathbf{s})$ that minimizes $\mathcal{E}(\mathbf{v})$ must satisfy the Euler-Lagrange (E-L) equations, which produce a set of d decoupled partial differential equations, $\mu(\mathbf{s})\partial_{tt}\mathbf{v}(\mathbf{s},t) + \gamma(\mathbf{s})\partial_t\mathbf{v}(\mathbf{s},t) - \nabla\cdot\big(\alpha(\mathbf{s})\nabla\mathbf{v}(\mathbf{s},t)\big) + \Delta\big(\beta(\mathbf{s})\Delta\mathbf{v}(\mathbf{s},t)\big) = \mathbf{q}\big(\mathbf{v}(\mathbf{s},t)\big)$ where $\mathbf{q}$, μ and γ represent the external forces, and the mass and the damping density of the model respectively.

By applying a spatial discretization by means of finite elements $\mathbf{v} = \mathbf{f} \circledast \mathbf{u}$, and using B-splines as shape function $\mathbf{f}$, the Garlekin's method allow us to transform the E-L equations of motion in a set of d second-order partial differential equations (PDE), $\mathbf{M}\mathrm{d}_{tt}\mathbf{u}_i(t) + \mathbf{C}\mathrm{d}_t\mathbf{u}_i(t) + \mathbf{K}\mathbf{u}_i(t) = \mathbf{q}_i(t)$, where $\mathbf{u}_i$ are the nodes of the model for the dimension i reshaped to a column vector, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the matrices of the model and $\mathbf{q}_i$ represents the external forces.

For a practical implementation, the time is discretized $\mathbf{u}_\xi = \mathbf{u}(\xi\Delta t)$, and the time derivatives of $\mathbf{u}$ are replaced by their discrete approximations. Thus, the system can be rewritten as, $\eta^{-1}(\eta\mathbf{f} + \mathbf{k}) \circledast \mathbf{u}_\xi = a_1\,\mathbf{f} \circledast \mathbf{u}_{\xi-1} + a_2\,\mathbf{f} \circledast \mathbf{u}_{\xi-2} + \eta^{-1}\mathbf{q}_{\xi-1}$, where $\circledast$ indicates e-dimensional circular convolution. η, γ, a_1 and a_2 are constants obtained from the model parameters.

The discrete spatial domain $\overline{\mathbf{n}}$ is translated into the frequency domain $\overline{\omega}$. This allows us to isolate the nodes, $\hat{\mathbf{u}}_\xi = \hat{\mathbf{h}}(a_1\hat{\mathbf{u}}_{\xi-1} + a_2\,\hat{\mathbf{u}}_{\xi-2} + (\eta\hat{\mathbf{f}})^{-1}\hat{\mathbf{q}}_{\xi-1})$ where $\hat{\mathbf{u}}$ and $\hat{\mathbf{q}}$ are the DFT's of their respective spatial sequences. $\hat{\mathbf{h}} = 1/(1 + \hat{\mathbf{k}}/(\eta\hat{\mathbf{f}}))$ is an e-dimensional low-pass filter, inverse of the high-pass filter $\hat{\mathbf{k}}$. This equation provides an efficient formulation because it is applied to each frequency component of the model independently.

To evaluate the stability and convergence conditions towards the final situation, a reduction of order in the system is applied. By defining $\hat{\mathbf{U}}_\xi = [\hat{\mathbf{u}}_\xi\ \hat{\mathbf{u}}_\xi]^\top$, $\hat{\mathbf{H}} = [a_1\hat{\mathbf{h}}\ \ a_2\hat{\mathbf{h}}\ ;\ 1\ \ 0]$ and $\hat{\mathbf{Q}}_{\xi-1} = [\hat{\mathbf{q}}_{\xi-1}/\eta\hat{\mathbf{f}}\ \ 0]^\top$, the system equation can be collected as $\hat{\mathbf{U}}_\xi = \hat{\mathbf{H}}_{\xi-1}\cdot\hat{\mathbf{U}}_\xi + \hat{\mathbf{Q}}_{\xi-1}$. Then, we study the the residual error $\hat{\mathbf{E}}_\xi = \hat{\mathbf{U}}_\xi - \hat{\mathbf{U}}$, assuming that $\hat{\mathbf{Q}}_{\xi-1} = \hat{\mathbf{Q}}$ and that $\hat{\mathbf{U}}_\xi \to \hat{\mathbf{U}}_\infty = \hat{\mathbf{U}}_\xi$. Thereby, $\hat{\mathbf{E}}_\xi$ can be calculated as $\hat{\mathbf{E}}_\xi = \hat{\mathbf{H}}\cdot\hat{\mathbf{E}}_{\xi-1}$. The matrix $\hat{\mathbf{H}}$ is diagonalized using its decomposition $\hat{\mathbf{H}} = \hat{\mathbf{L}}\hat{\mathbf{D}}\hat{\mathbf{L}}^{-1}$ where $\hat{\mathbf{D}}$ are the diagonal matrix with the eigenvalues $\hat{\lambda}_{1,2} = \frac{a_1\hat{\mathbf{h}}}{2} \pm \frac{1}{2}\sqrt{a_1^2\hat{\mathbf{h}}^2 + 4a_2\hat{\mathbf{h}}}$. By applying the Z transform it is simple to deduce that these eigenvalues match the poles of the system for each frequency component $\overline{\omega}$, $\hat{p}_{1,2} = \hat{\lambda}_{1,2}$.

Since iterative methods converge if and only if the spectral radius of the iteration matrix is strictly less than the unity, we have to ensure that $|\hat{p}_{1,2}| < 1\ \forall\overline{\omega}$. On the other hand, depending on the sign of the discriminant $\Delta = a_1^2\hat{\mathbf{H}}^2 + 4a_2\hat{\mathbf{H}}$, the resulting convergence of each frequency component can be underdamped ($\Delta < 0$), critically damped ($\Delta = 0$) of overdamped ($\Delta > 0$). The slowest convergence mode corresponds to frequency component with the largest pole in absolute value. So, the convergence can be optimized for a specific spectrum band, setting the parameters of the model for these frequencies as critically damped mode. Given certain values for α, β, μ and $\overline{\omega}$, the parameter $\hat{\gamma}_c$ that provides critically damped mode is $\hat{\gamma}_c = -\frac{2}{\hat{\mathbf{h}}}(\hat{\mathbf{h}} - 1 - \sqrt{1-\hat{\mathbf{h}}})$.

These results allow the use of the d-dimensional deformable models in the characterization of multidimensional data. As the described formulation allows an iterative process for each frequency component independently, this system can be applied efficiently to models of any dimension. Moreover, the convergence results support the optimization of the iterative process based on the frequency band of the multimensional data to be characterized.