Local and semilocal convergence analysis in Banach spaces of a seventh order family

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Abstract

In mathematical modelling, we often find the problem of solving the equation $F(x) = 0$, where F is an operator defined on some open set $\Omega \subset X$, $F : \Omega \to Y$ and X, Y are Banach spaces. Iterative methods are used to solve these problems, such as Newton's method, an optimal method of order two. However, the implementation of this method requires calculating the linear inverse operator $F'(x_n)^{-1}$ per iteration, which is computationally expensive. Therefore, it is common to replace this operator by divided differences of order one, $[x_n + F(x_n), x_n; F]$, obtaining Steffensen's method.

Due to the development of technology, it is necessary to design processes with higher orders of convergence that are also efficient. Therefore, we are interested in studying the following family of derivative-free iterative methods, described in [1],

$$\begin{cases} x_0 \text{ given in } \Omega, , \\ w_n = x_n + \gamma F(x_n), \\ y_n = x_n - [w_n, x_n; F]^{-1} F(x_n), \\ \mu_n = I - [w_n, x_n; F]^{-1}[y_n, w_n; F], \\ z_n = y_n - A(\mu_n)[y_n, x_n; F]^{-1} F(y_n), \\ \delta_n = I - [w_n, x_n; F]^{-1}[z_n, y_n; F] A(\mu_n), \\ x_{n+1} = z_n - D(\mu_n, \delta_n)[z_n, y_n; F]^{-1} F(z_n), \quad n \geq 0, \end{cases} \tag{1.1}$$

where $\gamma \in \mathbb{R}$ and $A, D : \mathcal{L}(X) \to \mathcal{L}(X)$ are linear weight operators, where $\mathcal{L}(X)$ is the domain of bounded linear operators, being $X = \mathbb{R}^n$ in this paper. This process is efficient and with order of convergence 7. Moreover, taking the real parameter γ as operators of the form $\gamma_n : \Omega \times \Omega \to \mathcal{L}(Y, X)$, the authors obtain families of methods with memory whose orders range from 7 to 9.21699.

The motivation for the study of these methods is their extension to operators in Banach spaces, which is the main result of our paper. In addition, we develop a study of the local and semilocal convergence of the method (1.1), using generalised (ω-continuity) conditions, which can be extended to methods with memory. This analysis is very useful because it helps to find initial points x_0 that guarantee the convergence of the family of methods.

References

[1] Alicia Cordero, Neus Garrido, Juan Ramon Torregrosa, Paula Triguero-Navarro. *Design of iterative methods with memory for solving nonlinear systems.* Authorea. September 05, 2022.

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