Optimal combinations of derivative securities

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1 Introduction

This paper combines the classical derivative stochastic pricing models, usual in Mathematical Finance, with the coherent [Artzner *et al.,* 1999] and expectation bounded [Rockafellar *et al.,* 2006] risk measures. In particular, we focus on both, the design of efficient investment strategies in a risk/return approach and the "inconsistencies" pointed out by several authors about the existence of sequences of investment strategies whose couple $(return, risk)$ tends to $(+\infty, -\infty)$. For instance, [Balbás *et al.,* 2016] proved that such existence holds even in ambiguous frameworks, that is, approaches showing uncertainty about the probabilities of the states of nature, and [Biagini and Pinar, 2013] showed the permanence of "inconsistencies" if the risk measure is replaced by a performance measure such as the gain-loss ratio of [Bernardo and Ledoit, 2000]. An important innovation of our study is the consideration of the natural constraints imposed by the order book of the financial market. It is known that the order book may impose several frictions (or transaction costs) related to commissions and bid-ask spreads, as well as limits to the size of the investment related to the market debt. Under these constraints, we will show that the "pathological equality $(return, risk) = (+\infty, -\infty)$" does not apply, but one can still create very high $return/risk$ ratios, *i.e.*, good deals, according to the terminology of [Cochrane and Saa-Requejo, 2000]. Empirical implementations of our theoretical results will be implemented in the Spanish derivative market, where it will be illustrated how the theory applies for two important risk measures, namely, the expected shortfall (or conditional value at risk) of [Rockafellar *et al.,* 1999] and the expectile of [Newey and Powell, 2987]. or [Bellini and Di Bernardino, 2017].

2 Methods

We present both scalar and vector optimization problems looking for the efficient strategies in a derivative market and under a risk/return framework. Since the involved risk measure ρ is both coherent and expectation bounded, its sub-gradient at zero (see [Rockafellar *et al.,* 2006], or [Kupper and Schachermayer, 2009] for an alternative discussion) plays a critical role in all

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of our portfolio choice problems. Dealing with important results of Convex Analysis, we show that there is a dual linear problem which is a Goal-Programming problem (see [Tamiz *et al.,* 1998]). The dual of this dual, that is, the bidual of the portfolio selection problem, becomes linear, and its solution provides us with the optimal investment strategies we were looking for. In other words, the optimal portfolio is found by solving a linear programming problem involving infinite-dimensional Hilbert spaces of random variables.

The vector optimization problem is the most general one. Three involved objectives are simultaneously optimized, namely, the portfolio price, its expected pay-off and its coherent risk. The existence of constraints provoked by the order book makes it rather difficult to represent the return in a simple mathematical expression, and that is the reason why the expected return is replaced by two objectives, namely, the portfolio price and its expected pay-off. If possible, it is worth simplifying the computational costs related to the resulting infinite-dimensional linear problem, and for this reason we propose an alternative scalar optimization problem. This second (simplified) problem only applies if the order book restrictions do not totally impede the ability to take advantage of the "theoretical pathology" $(return, risk) = (+\infty, -\infty)$. Nevertheless, when the scalar problem may be used, the related algorithms significantly accelerates the resolution process.

Both the scalar and the vector problem are given below. Further details may be found in [Balbás *et al.,* 2022] (vector problem) and [Balbás *et al.,* 2023] (scalar one). The vector problem becomes

$$\begin{cases} Min\ \Pi(x),\ Max\ \mathbb{E}\left(\sum_{j=1}^{n} x_j S_j\right),\ Min\ \rho\left(\sum_{j=1}^{n} x_j S_j\right) \\ \\ -c_j \leq x_j \leq C_j,\ j=1,2,...,n \end{cases} \quad (1)$$

where $(x_j)_{j=1}^n$ is the decision variable (the portfolio one is looking for), $\Pi(x)$ is its price, $(S_j)_{j=1}^n$ is a vector of random variables representing pay-offs at the investor planning period, $(c_j)_{j=1}^n$ and $(C_j)_{j=1}^n$ are limits to buy and sell the corresponding security, and $\mathbb{E}$ and ρ represent "mathematical expectation" and "risk", respectively. Its goal-programming dual becomes

$$\begin{cases} Min\ \sum_{j=1}^{n} c_j d_j^- + \sum_{j=1}^{n} C_j d_j^+ \\ \\ \lambda_j - \mathbb{E}\left(\left(\beta + \alpha z_\rho\right) S_j\right) - d_j^- + d_j^+ = 0,\ j=1,2,...,n \\ b_j \leq \lambda_j \leq a_j,\ j=1,2,...,n \\ z_\rho \in \Delta_\rho \\ d_j^+, d_j^- \geq 0,\ j=1,2,...,n \end{cases} \quad (2)$$

where $(\lambda_j)_{j=1}^n$ is a decision variable representing "pricing rule" ([Balbás *et al.,* 2022]), Δ_ρ is the risk measure sub-gradient, and $\left(\left((\lambda_j)_{j=1}^n\right)\left(\left(d_j^-\right)_{j=1}^n\right)\left(\left(d_j^+\right)_{j=1}^n\right), z_\rho\right)$ is the decision variable. Besides, the scalar problem and its dual become

$$\begin{cases} Min\ \rho\left(x_0 + \sum_{j=1}^n \left(x_j s_j - y_j S_j\right)\right) \\ x_0 + \Pi(x) \leq 0 \\ -c_j \leq x_j \leq C_j,\ j=1,2,...,n \end{cases} \quad (3)$$

and

$$\begin{cases} Min\ \sum_{j=1}^{n}\left(C_j d_j^{+} + c_j d_j^{-}\right) \\ q_j \leq \mathbb{E}\left(S_j z_\rho\right) - d_j^{+} \leq Q_j, \qquad j = 1, 2, ..., n \\ d_j^{-}, d_j^{+} \geq 0, \qquad j = 1, 2, ..., n \end{cases} \tag{4}$$

where $(q_j)_{j=1}^{n}$ represents bid prices and $(Q_j)_{j=1}^{n}$ represents ask prices.

3 Results

As said above, Problems (2), (4) and their duals were used in order to solve (1) and (3) in practice and implement real data linked studies related to the Spanish derivative market (see https://www.meff.es/ing/Home). Further details about the obtained results may be found in [Balbás *et al.,* 2022 and 2023]. In particular, during the two analyzed months, $November\_2018$ and $January\_2024$, almost every week one was able to find a combination of derivatives on the underlying Spanish index $IBEX\_35$ with negative price (the price of the sold securities was higher than the price of the purchased ones) and negative risk, where the risk is measured by the expected shortfall with a very high confidence level of the expectile (or both). If both values are negative, only the market debt may prevent the construction of the sequence related to the anomaly $(return, risk) = (+\infty, -\infty)$, but such strategies with two negative values enable investors to beat alternative portfolios available in alternative financial markets.

4 Conclusions

The combination of the usual stochastic pricing models of Mathematical Finance with downside risk measures or performance measures opens new opportunities in the exploration of portfolio choice problems, even more so if one bears in mind that this combination provokes theoretical anomalies or pathologies that should be analyzed in future studies. We have added the market imperfections provoked by the market order book (commissions, bid-ask spread, depth constraints, etc.) and have provided efficient and tractable optimization problems and algorithms to construct efficient strategies and analyze the empirical effects of the mentioned anomalies. Empirical analyses have been implemented in the Spanish derivative markets, and the obtained results confirm the potential benefits that investors may reach when dealing with all of these topics.

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