A complete probabilistic study to the Random Fractional Legendre Differential Equation: Solution, Moments and Density

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Abstract

The Legendre differential equation is widely utilized in physics and engineering, particularly in the domains of spherical harmonics, potential theory, and quantum mechanics. Its solutions, referred to as Legendre polynomials, play a crucial role in numerous scientific and engineering applications. For instance, they facilitate the resolution of the Laplace Partial Differential Equation (PDE) in spherical harmonics through the employment of variable separation in spherical coordinates.

Nowadays, the incorporation of uncertainty and fractional derivatives in Ordinary Differential Equations (ODEs) is a topic of interest. Its reason is because they allow us to obtain a more accurate description of the real problem to be modelled.

In this study, we extend the Legendre differential equation [1] to both fractional and randomized frameworks. On one hand, we replace classical derivatives with Caputo fractional operators. On the other hand, we treat the parameters within the differential equations as random variables rather than deterministic values. In summary, we define the random fractional Legendre differential equation as:

$$\begin{aligned} (1-t^2)({}^C D_0^{2\alpha} Y)(t) - 2t^{\alpha}({}^C D_0^{\alpha} Y)(t) + LY(t) &= 0, \qquad \alpha \in]0,1] \\ Y(0) &= C_0, \\ ({}^C D_0^{\alpha} Y)(0) &= \Gamma(\alpha+1) C_1, \end{aligned} \tag{1}$$

where L, C_1 and C_0 second order random variables and $({}^C D_0^{\alpha} Y)(t)$ represents the Caputo derivative.

The study we propose will be conducted in mean square sense. This approach allow us to extend the classical deterministic Newton Leibnitz theory into stochastic processes. We will construct a mean square convergent solution applying a generalization of the Random Frobenius method. Then, we will compute approximations of the two first statistical moments. At this point, we will realize that computing higher order moments could be complex and other techniques are required. Applying Random Transformation technique we will compute convergent approximations for the first probability density function, that they allow us to approximate higher order moments in a more straightforward manner.

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To conclude this study, we will show numerical examples where the convergence of the statistical moments and the first probability density function are appreciated.

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References

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