Global convergence of a fifth order iterative method under weaker conditions

Sukhjit Singh$^{\flat}$,[1] Sonia Yadav $^{\flat}$ and Eulalia Martínez$^{\natural}$ and M.A. Hernández-Verón $^{\natural\natural}$

($\flat$) Department of Mathematics and Computing,
Dr B. R. Ambedkar National Institute of Technology, Jalandhar, India.

($\natural$) Instituto Universitario de Matemática Multidisciplinar,
Universitat Politècnica de València. València, Spain.

($\natural\natural$) Department of Mathematics and Computation,
University of La Rioja, Spain

1 Introduction

In Science and Engineering, many challenging problems require the solution of nonlinear equations of the form

$$\mathcal{R}(p) = 0, \tag{1}$$

where $\mathcal{R} : \Omega \subset \mathcal{P} \to \mathcal{Q}$ be continuous Fréchet differentiable operator in an open convex domain Ω of the Banach space $\mathcal{P}$ to $\mathcal{Q}$. This equation can represented in the form of an integral equation, differential equation, or a system of nonlinear equations (see [1]). Since exact solutions to nonlinear equations are rarely found in the literature, so we apply iterative methods to approximate their solutions. One common approach is the fixed-point method [3] in which we rewrite equation (1) as $p = \mathcal{H}(p)$ with $\mathcal{H}(p) = p - \mathcal{R}(p)$. If $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ is a contraction in convex and compact subset $\mathcal{G}$ of Banach space $\mathcal{P}$, then the sequence generated by

$$p_{k+1} = \mathcal{R}(p_k), k \geq 0$$

converges to a unique fixed point. This approach possesses two remarkable characteristics: it exhibits global convergence within $\mathcal{G}$ and ensures the presence of a fixed point. We can not achieve both properties simultaneously in any other iterative method but fixed point method offers a linear convergence rate. The most popular quadratically convergent Newton's method

$$p_0 \in \Omega, \qquad p_k = p_{k-1} - [\mathcal{R}'(p_{k-1})]^{-1}\mathcal{R}(p_{k-1}), \quad k \in \mathbb{N},$$

and its variants [8, 11] are applied for approximating the solution because of its low computational cost. Notice that we generally analyze the convergence in two ways : local [10] and semilocal convergence [9]. In local convergence, we require condition on p^*, the operator $\mathcal{R}$ and obtain the convergence ball $\overline{B(p^*, R)}$ in which the sequence converges starting from any point in the domain. On the other hand, in semilocal convergence, we require condition on p_0, the operator $\mathcal{R}$ and

[1] sukhjitmath@gmail.com

obtain the existence ball $B(p_0, R)$ which guarantee the existence of p^* in the ball. Arroyo *et al.* [2] presented convergence analysis of fifth order iterative method given by

$$\begin{cases} p_0 \text{ given in } \Omega, \\ q_k = p_k - \Lambda_k \mathcal{R}(p_k), \\ r_k = q_k - 5\Lambda_k \mathcal{R}(q_k), \\ p_{k+1} = r_k - \frac{1}{5}\Lambda_k(-16\mathcal{R}(q_k) + \mathcal{R}(r_k)), \quad k \geq 0, \end{cases} \tag{2}$$

where $\Lambda_k = [\mathcal{R}'(p_k)]^{-1}$. The beauty of method (2) is that it requires only first order Frèchet derivative same as Newton's method but having fifth order convergence.

Now a days, various authors combine local and semilocal convergence with the help of auxiliary point $\tilde{p}$. For this, we require the condition on auxiliary point (see [4, 5, 6]) and the operator $\mathcal{R}$ provides the existence ball as well as the domain in which sequence converges starting from any point in this domain. This is known as restricted global convergence. By using auxiliary points, Ezquerro and Hernández [4] obtained the global convergence domain for Newton's method, considering a Lipschitz condition on the first derivative. In [6], they further extended this approach for Chebyshev's method under Lipschitz condition on $\mathcal{R}$". Ezquerro et al. [7] established the domain of global convergence for the family (2) under Lipschitz condition on $\mathcal{R}'$. Yadav and Singh [12] extended the domain of global convergence of method (2) under the Lipschitz condition on $\mathcal{R}'$. The results established in [12] are not applicable for the following nonlinear Hammerstein type integral equations

$$p(\upsilon) + \lambda \sum_{i=1}^{m} \int_a^b K_i(\upsilon, \theta) M_i(p(\theta)) d\theta = f(\upsilon), \qquad \upsilon \in [a, b]$$

where, f, M_i, K_i for $i = 1, 2, ..., m$ are known and we have to determine a solution p^*. Equation (1) is equivalent to

$$G(p)(\upsilon) = p(\upsilon) + \lambda \sum_{i=1}^{n} \int_a^b K_i(\upsilon, \theta) M_i(p(\theta)) d\theta - f(\upsilon).$$

In this case, for each $y \in \Omega$, we have

$$[G'(p)y](\upsilon) = y(\upsilon) + \lambda \sum_{j=1}^{n} \int_a^b K(\upsilon, \theta) M_j'(p(\theta)) y(\theta) d\theta = y(\upsilon) + \lambda \int_a^b K(\upsilon, \theta) \sum_{j=1}^{n} M_j'(p(\theta)) y(\theta) d\theta$$

and, if each M_j' is Hölder continuous, then using max-norm, we can obtain

$$\|G'(p) - G'(q)\| \leq \sum_{j=1}^{n} N_j \|p - q\|^{\nu_j}, \quad N_j \geq 0, \nu_j \in [0, 1], \text{ for all } p, q \in \Omega.$$

Although the Lipschitz and Hölder's conditions do not hold, the ω-condition is satisfied.

Our main objective is to conduct a global convergence analysis of the fifth-order iterative method (2) under the ω-condition on $\mathcal{R}'$. Our research holds significance as it addresses scenarios where the Lipschitz condition may not hold, but the ω-condition is satisfied. Additionally, we explore theorem of existence and uniqueness, as well as the domain of global convergence for the solution. In order to establish the convergence theorems, we will construct the following Lemma's.

Lemma 1. *Consider $\mathcal{R} : \Omega \subseteq \mathcal{P} \to \mathcal{Q}$ be a continuous twice differentiable operator of the Banach space $\mathcal{P}$ to $\mathcal{Q}$. In consequence, we obtain*

(i) $\mathcal{R}(p_0) = \mathcal{R}(\tilde{p}) + \mathcal{R}'(\tilde{p})(p_0 - \tilde{p}) + \int_{\tilde{p}}^{p_0} (\mathcal{R}'(p) - \mathcal{R}'(\tilde{p})) dp$, *with $p_0 \in \Omega$.*

(ii) $\mathcal{R}(q_k) = \int\limits_0^1 (\mathcal{R}'(p_k + \tau(q_k - p_k)) - \mathcal{R}'(p_k))(q_k - p_k)d\tau$, *with* $p_k, q_k \in \Omega$.

(iii) *For* $q_k, p_{k+1} \in \Omega$, *it follows*
$\mathcal{R}(p_{k+1}) = \mathcal{R}(q_k) + \mathcal{R}'(q_k)(p_{k+1} - q_k) + \int\limits_{q_k}^{p_{k+1}} (\mathcal{R}'(p) - \mathcal{R}'(q_k))dp.$

(iv) *For* $p_k, q_k, r_k \in \Omega$, *it follows*

$$\frac{9}{5}\mathcal{R}(q_k) + \frac{1}{5}\mathcal{R}(r_k) = \frac{4}{5}\int_0^1 (\mathcal{R}'(p_k + \tau(q_k - p_k)) - \mathcal{R}'(p_k))(q_k - p_k)d\tau + \frac{1}{5}\int_0^1 (\mathcal{R}'(p_k + \tau(r_k - p_k)) - \mathcal{R}'(p_k))(r_k - p_k)d\tau.$$

(v) *For* $p_k \in \Omega$, *it follows*

$$\mathcal{R}(p_k) + (\tilde{p} - p_k)\mathcal{R}'(p_k) = \mathcal{R}(\tilde{p}) - \int_{p_k}^{\tilde{p}} (\mathcal{R}'(p) - \mathcal{R}'(p_k))dp.$$

Furthermore, let's assume $\exists\ \rho > 0$ such that $p \in B(\tilde{p}, \rho) \subset \Omega$, and

$$\beta\tilde{\omega}(\rho) < 1. \tag{3}$$

Let $b_0 = \dfrac{\beta\omega(\mu)}{1 - \beta\tilde{\omega}(\rho)}$ and define a real positive sequence $\{b_k\}$ such that

$$b_{k+1} = \phi(b_k)^{\upsilon} b_k, \qquad k \geq 0, \tag{4}$$

$$\phi(p) = \left(\frac{p}{1+\upsilon} + (1+p)\xi(p)\big(\frac{p}{1+\upsilon}\xi(p)^{1+\upsilon}\big)\right) \tag{5}$$

and

$$\xi(p) = \left(\frac{4p}{5(1+\upsilon)} + \frac{p}{5(1+\upsilon)}\big(1 + \frac{5p}{1+\upsilon}\big)^{1+\upsilon}\right) \tag{6}$$

As, $\|I - \tilde{\Lambda}\mathcal{R}'(u)\| \leq \|\tilde{\Lambda}(\mathcal{R}'(\tilde{p}) - \mathcal{R}'(p))\| \leq \beta\tilde{\omega}(\rho) < 1$. Using Banach Lemma, we deduce

$$\|\Lambda\| = \|[\mathcal{R}'(p)]^{-1}\| \leq \frac{\beta}{1 - \beta\tilde{\omega}(\rho)} = d, \tag{7}$$

and

$$\|\Lambda\mathcal{R}'(\tilde{p})\| \leq \frac{1}{1 - \beta\tilde{\omega}(\rho)}. \tag{8}$$

Lemma 2. *For* $t > 0$, $\phi(t)$ *and* $\xi(t)$ *are increasing functions defined by* (5) *and* (6). *If* $b_0 < 0.2287$ *then* $\phi(b_0) < 1$, $\xi(b_0) < 2b_0$ *and the sequence* $\{b_k\}$ *is decreasing.*

Lemma 3. *For the real valued functions* $\phi(p)$ *and* $\xi(p)$ *defined by* (5) *and* (6), *if there exists* $\rho > 0$ *such that the condition* (3) *and*

$$\frac{5b_0\mu}{(1+\upsilon)} + \frac{(1+\upsilon)\eta + \beta\rho\omega(\rho)}{(1+\upsilon)(1 - \beta\tilde{\omega}(\rho))} \leq \rho. \tag{9}$$

satisfied, then the following recurrence relation true for every $k \geq 1$:

$$
\begin{aligned}
\|q_k - p_k\| &\leq \phi(b_k)\|q_{k-1} - p_{k-1}\|,\\
\|q_k - \tilde{p}\| &\leq \frac{(1+\upsilon)\eta + \beta\rho\omega(\rho)}{(1+\upsilon)(1-\beta\tilde{\omega}(\rho))},\\
\|r_k - q_k\| &\leq \frac{5}{(1+\upsilon)} b_k \|q_k - p_k\|,\\
\|r_k - \tilde{p}\| &\leq \frac{(1+\upsilon)\eta + \beta\rho\omega(\rho)}{(1+\upsilon)(1-\beta\tilde{\omega}(\rho))} + \frac{5}{(1+\upsilon)} b_k \mu,\\
\|p_{k+1} - p_k\| &\leq (1+\xi(b_k))\|q_k - p_k\|,\\
\|p_{k+1} - \tilde{p}\| &\leq \frac{(1+\upsilon)\eta + \beta\rho\omega(\rho)}{(1+\upsilon)(1-\beta\tilde{\omega}(\rho))} + \xi(b_k)\left(\prod_{i=0}^{n-1}\phi(b_i)\right)\mu.
\end{aligned}
\tag{10}
$$

Theorem 1. *For some $\tilde{p} \in \Omega$, assume $\exists\ \rho > 0$ satisfying the conditions (3) and (9) for $B(\tilde{p}, \rho) \subset \Omega$. If $b_0 < 0.2287$, the iterative method provided by (2) is well-defined and converges to p^* in $\overline{B(\tilde{p}, \rho)} \subset \Omega$ for each point $p_0 \in B(\tilde{p}, \rho)$.*

Theorem 2. *Under the conditions of Theorem 1, p^* is unique in $\overline{B(\tilde{p}, \rho^*)} \cap \Omega$, where ρ^* is a positive root of $\frac{2\beta}{1+\upsilon}\tilde{\omega}(\rho+\rho^*)\big(1 - \frac{1}{2^{(1+\upsilon)}}\big) = 1$.*

We will also verify the above theoretical results on the nonlinear Hammerstein type integral equations to show the applicabilty of our approach.

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