Local convergence analysis of a fourth order family of iterative methods in Banach spaces

Eulalia Martínez[♭], Arleen Ledesma[♮1]

(♭) I.U. de Matemática Multidisciplinar, Universitat Politècnica de València
Camí de Vera s/n, València, Spain.

(♮) Escuela de Matemática, Universidad Autónoma de Santo Domingo (UASD),
Alma Máter, 10105 Santo Domingo, Dominican Republic.

1 Introduction

Problem solving in science and engineering, through the process of mathematical modeling, is one major challenge in Numerical Functional Analysis. Under some assumptions, a particular problem is modeled into a nonlinear equation in Banach spaces.

Let F be a nonlinear operator, $F : D \subseteq X \to Y$, where X, Y are Banach spaces and D, the domain of F, is an open convex subset of X. Obtaining the best approximation to the solution of an equation $F(x) = 0$ is always the main goal in the study of an iterative method, but it is of utmost importance to be certain that the chosen iterative method converges to this approximation. In order to achieve a good output value and convergence of the method to the solution, we can perform studies focused on the conditions that have to be fulfilled by the solution x^*, the initial value x_0 or the operator, using the iterative method.

In Banach spaces, local and semilocal convergence can be analyzed. When analyzing local convergence (see [1], [3]), conditions are imposed on the operators and their derivatives at the solution x^*. The result is the ball of local convergence, centered on the solution with radius r: $B(x^*, r)$. The elements within this ball are the potential initial estimates of the method from which we can guarantee convergence, that is, the possible starting points for the method to work or for which the limit of the iterates sequence is the solution of the problem. In addition, the error bound is obtained.

For the semilocal convergence analysis, conditions are imposed on the operators and their derivatives at the initial estimate x_0 and iterations are performed. As a result we obtain the existence and uniqueness of the solution, the R-order of convergence of the method, the a priori error bounds and the convergence domain (within which the operator F is defined).

In this paper, we describe an iterative method and its local convergence analysis to solve the nonlinear equation $F(x) = 0$ with solution x^*. Consider the family of iterative methods [2], defined

[1] aledesma73@uasd.edu.do

for $k = 0, 1, 2, \ldots$ by

$$
\begin{aligned}
y_k &= x_k - \frac{2}{3} F'\left(x_k\right)^{-1} F\left(x_k\right) \\
x_{k+1} &= x_k - \left[\left(\frac{5}{8} - \alpha\right) + \alpha F'\left(y_k\right)^{-1} F'\left(x_k\right) + \frac{\alpha}{3} F'\left(x_k\right)^{-1} F'\left(y_k\right)\right. \\
&\quad \left. + \left(\frac{3}{8} - \frac{\alpha}{3}\right)\left(F'\left(y_k\right)^{-1} F'\left(x_k\right)\right)^2\right] F'\left(x_k\right)^{-1} F\left(x_k\right).
\end{aligned}
\tag{1}
$$

where parameter $\alpha \in \mathbb{C}$ and x_0 is the initial estimate. The order of convergence of the scheme is 4.

We wish to solve the nonlinear equation $F(x) = 0$. It is assumed the existence of $F'(x)$ in a neighborhood of the solution x^* and $F'\left(x^*\right)^{-1} \in BL(Y, X)$, where $BL(Y, X)$ is the set of bounded linear operators from Y to X.

2 Local convergence analysis

Cordero et al in [2], studied scheme (1) using complex dynamics tools, but a theoretical study of local convergence was not carried out, which we consider quite important for the aforementioned reasons. Next, we will proceed with this analysis.

2.1 Local convergence analysis using conditions (C_1), (C_2) and (C_3)

Let $B(w, \rho)$ y $\overline{B}(w, \rho)$ be the open and closed balls, respectively, in X, centered in w with radius $\rho > 0$. For the local convergence analysis of (1), the following conditions are imposed on operators $(F,\, F')$ in the solution x^*, which is assumed to exist.

For the local convergence analysis, we assume the following conditions for real numbers $L_0 > 0$, $L > 0$ and for all $x,\, y \in D$:

(C_1) $F\left(x^*\right) = 0, \quad F'\left(x^*\right)^{-1} \in BL(Y, X)$

(C_2) $\left\|F'\left(x^*\right)^{-1}\left(F'(x) - F'\left(x^*\right)\right)\right\| \leqslant L_0\left\|x - x^*\right\|$

(C_3) $\left\|F'\left(x^*\right)^{-1}\left(F'(x) - F'(y)\right)\right\| \leq L\|x - y\|$

(C_4) $\left\|F'\left(x^*\right)^{-1} F(x)\right\| \leqslant M,\ \forall x \in D$ for $M \geqslant 1$ real.

In order to perform the local convergence analysis of the method, it is suitable to define certain functions and parameters. Define function g_1 on the interval $\left[0, \frac{1}{L0}\right[$ by

$$
g_1(t) = \frac{1}{1 - L_0 t}\left[\frac{L}{2} t + \frac{1}{3}\left(1 + L_0 t\right)\right] \tag{2}
$$

Now, consider the function $h_1(t) = g_1(t) - 1$, then $h_1(0) = 1/3 - 1 = -2/3 < 0$ and $h_1\left(1/L_0\right) = +\infty$. Consequently, $h_1(t)$ has at least one root in $]0, 1/L_0[$ by the intermediate value theorem. Let r_1 the smallest root in $]0, 1/Lo[$, it follows that

$$
0 < r_1 < 1/L_0 \quad \text{y} \quad 0 \leq g_1(t) < 1, \ \ \forall t \in [0, r_1[. \tag{3}
$$

Moreover, define function g_2 in the interval $[0, r_1[$ by

$$g_2(t) = \frac{L}{2}t + \left|\frac{3}{8}+\alpha\right|\frac{1+L_0\,t}{1-L_0\,t} + |\alpha|\,\frac{1+L_0\,t}{1-L_0\,t\,g_1(t)} + \left|\frac{\alpha}{3}\right|\frac{1+L_0\,t\,g_1(t)}{1-L_0\,t}(1+L_0\,t)$$
$$+\left|\frac{3}{8}-\frac{\alpha}{3}\right|\left(\frac{1+L_0\,t}{1-L_0\,t\,g_1(t)}\right)^2 t.$$

Consider the function $h_2(t) = g_2(t) - 1$. Following that $g_1(0) = \frac{1}{3}$, $g_2(0) = \left|\frac{3}{8}+\alpha\right| + |\alpha| + \left|\frac{\alpha}{3}\right|$, then $h_2(0) = \left|\frac{3}{8}+\alpha\right| + |\alpha| + \left|\frac{\alpha}{3}\right| - 1$ and $h_2(0) < 0$ if $\alpha \in \left[-\frac{33}{56}, \frac{15}{56}\right]$, also

$$h_2(r_1) = \frac{L}{2}r_1 + \left|\frac{3}{8}+\alpha\right|\frac{1+L_0\,r_1}{1-L_0\,r_1} + |\alpha|\,\frac{1+L_0\,r_1}{1-L_0\,r_1\,g_1(r_1)} + \left|\frac{\alpha}{3}\right|\frac{1+L_0\,r_1\,g_1(r_1)}{1-L_0\,r_1}(1+L_0\,r_1)$$
$$+\left|\frac{3}{8}-\frac{\alpha}{3}\right|\left(\frac{1+L_0\,r_1}{1-L_0\,r_1\,g_1(r_1)}\right)^2 r_1 - 1.$$

From (3), we have that $1 + L_0\,r_1 > 0$ and $1 - L_0\,r_1\,g_1 > 0$, and conclude that $h_2(r_1) > 0$.

Consequently, $h_2(t)$ has at least one root in $]0, r_1[$. Let r be the smallest root in $]0, r_1[$, we have

$$0 < r < r_1 < 1/L_0 \quad \text{y} \quad 0 \le g_2(t) < 1, \ \forall t \in [0, r[. \tag{4}$$

Thus, for $\alpha \in \left]-\frac{33}{56}, \frac{15}{56}\right[$, $0 < r < r_1 < 1/L_0$.

Let us consider the following lemma:

Lemma 2.1 *If operator F satisfies (C_2) and (C_3), then the following inequalities hold, for all $x \in D$ and $t \in [0,1]$,*

$$\begin{aligned}
&\left\|F'(x^*)^{-1}F'(x)\right\| \leqslant 1 + L_0\,\|x - x^*\| \\
&\|F'(x^*)^{-1}F'(x^* + t(x - x^*))\| \le 1 + L_0\|\,x - x^*\| \\
&\left\|F'(x^*)^{-1}F(x)\right\| \le (1 + L_0\,\|x - x^*\|)\,\|x - x^*\|
\end{aligned} \tag{5}$$

Next, we provide the local convergence result for method (1), given the (C_1) - (C_4) conditions. For Theorem (2.2), condition (C_4) is dropped and the radius of the convergence ball is obtained without using constant M.

Theorem 2.2 *Let $F : D \subseteq X \to Y$ a differentiable Fréchet operator. Suppose that there exists $x^* \in D$ y $\alpha \in \left]-\frac{33}{56}, \frac{15}{56}\right[$ such that (C_2) and (C_3) are satisfied and $\overline{B}(x^*, r) \subseteq D$, where r is the radius. The sequence x^* generated by (1) for $x_0 \in B(x^*, r) - \{x^*\}$ is well defined for $k = 0, 1, 2, \ldots$ remains in $B(x^*, r)$ and converges to x^*. In addition, the following estimates hold $k = 0, 1, 2, \ldots$:*

$$\|y_k - x^*\| \le g_1(\|x_k - x^*\|)\,\|x_k - x^*\| \le \|x_k - x^*\| \le r,$$

$$\|x_{k+1} - x^*\| \le g_2(\|x_k - x^*\|)\,\|x_k - x^*\| \le \|x_k - x^*\| \le r,$$

where the g functions are defined before Theorem (2.2). Furthermore, if $R \in \left[r, \frac{2}{L_0}\right[$ exists such that $\overline{B}(x^*, R) \subseteq D$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $\overline{B}(x^*, R)$.

2.2 Local convergence analysis using condition (C_4)

In the previous section we avoided the use of boundedness conditions, which sometimes is a disadvantage when solving practical problems. In this section, condition (C_4) is applied, allowing us to compare the radii obtained when using M with those obtained avoiding the use of M. The local convergence analysis is hereby completed.

Theorem 2.3 *Let $F : D \subseteq X \to Y$ de a differentiable Fréchet operator, $L_0 > 0, L > 0, M \geq 1$ and $\alpha \in \left]-\frac{33}{56}, \frac{15}{56}\right[$. Let $x^* \in D$ such that, for all $x, y \in D$, conditions (C_1) - (C_4) hold and, moreover, it is verified that $M < 3$ and $\left[\left|\frac{3}{8}+\alpha\right| + |\alpha| + \left|\frac{\alpha}{3}\right|\right] M < 1$. Then, the sequence x^* is well defined for $x_0 \in B(x^*, r) \subseteq D$, for $k = 0, 1, 2, \ldots$ and converges to x^*. In addition, the estimates comply, for each $k = 0, 1, 2, \ldots,$*

$$\|y_k - x^*\| \leq g_1\left(\|x_k - x^*\|\right)\|x_k - x^*\| \leq \|x_k - x^*\| \leq r,$$

$$\|x_{k+1} - x^*\| \leq g_2\left(\|x_k - x^*\|\right)\|x_k - x^*\| \leq \|x_k - x^*\| \leq r.$$

Furthermore, if there exists $R \in \left[r, \frac{2}{L_0}\right[$ such that $\overline{B}(x^*, R) \subseteq D$, then the limit point x^* is a unique solution of the equation $F(x) = 0$ in $\overline{B}(x^*, R)$.

3 Conclusions

In this paper we have established local convergence analysis for a family of iterative methods in Banach spaces under the assumption that the Fréchet derivative satisfies the Lipschitz continuity condition. The radii of balls of convergence have been obtained.

Acknowledgments

This study has been funded by: Ayuda a Primeros proyectos de Investigación (on PAID-06-23), Vicerrectorado de Investigación de la Universitat Politècnica de València, (UPV), in the framework of project MERLIN.

References

[1] Argyros, J., Cho, Y., George, S., Local convergence for some third-order iterative methods under weak conditions, *Journal of the Korean Mathematical Society*, Volume 53, Issue 4, 781–793, 2016.

[2] Cordero, A., Ledesma, A., Torregrosa, J., G. Maimó, J., Design and dynamical behavior of a fourth order family of iterative methods for solving nonlinear equations, *AIMS Mathematics*, Volume 9, Issue 4, 8564–8593, 2024.

[3] Martínez, E., Hueso, J., Singh, S., Gupta, D., Enlarging the convergence domain in local convergence studies for iterative methods in Banach spaces, *Applied Mathematics and Computation*, Volume 281, 252–265, 2016.