About an stochastic differential equation with nonlocal discrete diffusion modeling life tables

José Valero$^{\flat}$,[1] Tomás Caraballo$^{\natural}$ and Francisco Morillas^{c}
($\flat$) Centro de Investigación Operativa, Universidad Miguel Hernández de Elche
Avda. Universidad s/n, 0302-Elche, Spain.
($\natural$) Universidad de Sevilla, Departamento de Ecuaciones Diferenciales y Análisis Numérico
Apdo. de Correos 1160, 41080-Sevilla Spain
(c) Universitat de València, Departament d'Economia Aplicada, Facultat d'Economia,
Campus dels Tarongers s/n, 46022-València, Spain.

1 Introduction

First, we consider the following stochastic system of differential equations

$$\begin{aligned} &\frac{d}{dt}u_i\left(t\right) = \sum_{r\in D} j_{i-r}u_r\left(t\right) - u_i(t) + \sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r\left(t\right) + b\sigma_i(u_i(t))\frac{dw_i}{dt},\ i\in D,\ t>0, \\ &u_i\left(0\right) = u_i^0,\ i\in D, \end{aligned} \tag{1}$$

where $w_i\left(t\right)$ are independent Brownian motions and $b>0$ is the intensity of the white noise. We will assume the following assumptions on the kernel $\{j_i\}$ and the functions $\{g_i(\cdot)\}$:

$(H1)$ $j_i \geq 0$ for all $i\in\mathbb{Z}$.

$(H2)$ $\sum_{i\in\mathbb{Z}} j_i = 1$.

$(H3)$ $g\in C([0,+\infty), l_2^\infty)$,
where $l_2^\infty = \{(u_i)_{i\in\mathbb{Z}\setminus D} : \sup_{i\in\mathbb{Z}\setminus D}|u_i|\}$.

We take into account two specific type of noises. Namely: 1) $\sigma_i(v) = v$ (linear case); 2) $\sigma_i(v) = v(1-v)$. The choice of the noise in the second case is motivated by the fact that we are interested in studying variables like the probability of death, which take values in the interval $[0,1]$.

Second, we study the following stochastic system of differential equations with delay

$$\left\{ \begin{array}{c} \dfrac{d}{dt}u^i\left(t\right) = \left[J(t,u_t)\right]_i - u^i(t) + b\sigma_i(u_i(t))\dfrac{dw_i}{dt},\ i\in D,\ t>0, \\ u^i\left(\tau+s\right) \equiv \phi^i\left(s\right),\ i\in D,\ s\in[-h,0], \end{array}\right. \tag{2}$$

[1] correspondingauthor@mail.com

where $h > 0$, $w_i(t)$ are independent Brownian motions, $b \geq 0$ is the intensity of the white noise, $\sigma_i : \mathbb{R} \to \mathbb{R}$, and $J : \mathbb{R}^+ \times C([-h,0],\mathbb{R}^m) \to \mathbb{R}^m$ is the non-autonomous convolution operator defined by

$$[J(t,u_t)]_i = \sum_{r\in D} \int_{-h}^{0} j_{i-r} u^r(t+s)\alpha_i(s)d\mu(s) + \sum_{r\in \mathbb{Z}\backslash D} \int_{-h}^{0} j_{i-r} g_r\,(t+s)\,\alpha_i(s)d\mu(s), \text{ if } i \in D,$$

where $u(t) = \left(u^i(s)\right)_{i\in D}$, $u_t = u(t+s)$ for $s \in [-h,0]$, $j : \mathbb{Z} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{Z}\backslash D \to \mathbb{R}$ and $d\mu(s) = \xi(s)\,ds$ being $\xi(\cdot)$ a probability density. We assume the following conditions:

$(H1)$ $j_k \geq 0$ for all $k \in \mathbb{Z}$.

$(H2)$ $\sum_{i\in\mathbb{Z}} j_i = 1$.

$(H3)$ $g \in C([0,+\infty), l_2^\infty)$.

$(H4)$ $\alpha_i \in C([-h,0],\mathbb{R})$, $\alpha_i(s) \geq 0$ for $i \in D$, $s \in [-h,0]$.

We study the properties of solutions of these systems and perform some numerical simulations related to life tables.

2 Results

First, we obtain some theoretical results concerning the properties of solutions. In the linear case, we show the existence of a unique positive solution whenever the initial datum is positive. In the second case, we establish that if the initial condition takes values in the interval $(0,1)$, then the solution remains in this interval for any future time. Finally, we analyse the asymptotic behaviour of solutions, showing under certain assumptions that for large times they remain in a neighborhood of the unique fixed point of the deterministic system.

Second, we perform numerical simulations of solutions and analyse the efficiency of these models for prediction of mortality using data from Spain.

The results for problem (1) were published in [1].

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References

[1] T. Caraballo, F. Morillas, J. Valero, On a stochastic nonlocal system with discrete diffusion modeling life tables, *Stochastic and Dynamics*, V.22, N.7, 2022.