Inverse matrix estimations by iterative methods

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1 Introduction

Several iterative schemes have recently been designed to approximate the inverse of a nonsingular complex matrix of $n\times n$. These types of problems have applications in various fields of science and engineering. For additional references on these researches, see [2, 3].

In the literature, several iterative algorithms with memory developed to solve the scalar equation $f(x)=0$ have been extended to compute the inverse of a non-singular complex matrix A of size $n\times n$, i.e., the zero of the non-linear matrix equation $F(X)=X^{-1}-A=0$, where $F:\mathbb{C}^{n\times n}\rightarrow\mathbb{C}^{n\times n}$ is a non-linear matrix function.

Newton-Schulz and Chebyshev are two well-known iterative methods that allow us to approximate A^{-1} without using inverse operators in their application. The most common of these schemes is Newton-Schulz (see [3]), whose iterative formula is

$$X_{k+1}=X_k(2I-AX_k),\quad k=0,1,2\cdots, \tag{1}$$

where I is the identity matrix of $n\times n$.

In [7] Schulz proved that the scheme given in equation (1) converges if and only if the eigenvalues of the matrix $I-AX_0$ are lower than 1. At each iteration of the Newton-Schulz method the residuals $E_k=I-AX_k$ satisfy the inequality $\|E_{k+1}\|\leq\|E_{k+1}\|^2$ for all k, and therefore, scheme (1) converges quadratically.

Chebyshev's method was extended to the computation of inverse matrices by Amat et al. in [1], who obtained the following iterative expression free of inverse operators, with third order of convergence

$$X_{k+1}=3X_k-3X_kAX_k+X_kAX_kAX_k,\quad k=0,1,2\cdots, \tag{2}$$

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On the other hand, Li et al. in [6] studied the application of Homeier's method for estimating the inverse of a matrix, and proposed the following iterative scheme

$$X_{k+1} = X_k[I + \frac{1}{2}(I - AX_k)(I + (2I - AX_k)^2)],\ k = 0, 1, 2 \cdots, \tag{3}$$

Kansal et al. in [5] proved that this scheme converges cubically.

The main goal of this paper is to design a parametric family of iterative methods without memory to compute the inverse of a nonsingular matrix without using inverse operators in its iterative expression. We will demonstrate the order of convergence of the family and perform numerical tests to confirm the theoretical results obtained.

2 Parametric family under study

For a scalar equation $f(x) = 0$, let us consider the following family of fourth-order iterative schemes:

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)} \\ z_k &= y_k - \alpha \frac{f(y_k)}{f\left[x_k, y_k\right]} \\ x_{k+1} &= z_k - \frac{f(z_k)}{f\left[y_k, z_k\right]} \end{aligned} \tag{4}$$

where the parameter $\alpha \in \mathbb{R}$ and the divided difference $f\left[x_k, y_k\right]$ is defined as $f\left[x_k, y_k\right] = \dfrac{f(y_k) - f(x_k)}{y_k - x_k}$.

For a matrix equation $F(X) = X^{-1} - A$, the iterative expression of the family given in (4) can be expressed as:

$$X_{k+1} = X_k\left((4+\alpha)I + AX_k\left(-2I + AX_k\left(2\left(2+3\alpha\right) + AX_k\left(-I\left(1+4\alpha\right) + \alpha AX_k\right)\right)\right)\right), \quad k \geq 0. \tag{5}$$

The following result sets the order of convergence of the family of iterative schemes given in (5), which we designate as FM3.

Theorem 2.1 *Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular matrix. Let X_0 be an initial approximation such that $\|I - AX_0\| < 1$. If $\alpha \in [0, 1]$, then the sequence $\{X_k\}$, obtained by (5) converges to A^{-1} with convergence order $p = 4$ and satisfies the following error equation, where $e_k = X_k - A^{-1}$.*

$$\|e_{k+1}\| \leq \|A^3\|\ \|e_k\|^4.$$

3 Numerical Results

In this section, we show numerical tests of four different methods of the family FH3 given by the equation (5), designed to compute the inverse of a nonsingular matrix A of $n{\times}n$. The numerical tests were carried out in Matlab R2023b, using an Intel Core i7-1065G7 processor up to 3.9 GHz, 16 GB DDR4 RAM. The stopping criterion used is $\|X_{k+1} - X_k\|_2 < 10^{-6}$ or

Table 1: Results obtained by approximating the inverse of a random matrix of order $n = 50$.

Method	n	COC	it	$\| X_{k+1} - X_k \|_2$	$\| I - AX_k \|_2$
Block 1: Convergence conditions are met ($0 \leq \alpha \leq -1$)					
FM3 $\alpha = 1$	50	5.0000	10	2.77×10^{-02}	2.26×10^{-14}
FM3 $\alpha = 0.5$	50	2.5149	11	4.76×10^{-04}	2.57×10^{-14}
Block 2: Convergence conditions are not satisfied					
FM3 $\alpha = -1$	50	3.8046	14	5.21×10^{-03}	2.22×10^{-14}
FM3 $\alpha = 2$	50	3.6825	9	1.04×10^{-01}	5.85×10^{-10}

Table 2: Results obtained by approximating the inverse of a random matrix of order $n = 100$.

Method	n	COC	it	$\| X_{k+1} - X_k \|_2$	$\| I - AX_k \|_2$
Block 1: Convergence conditions are met ($0 \leq \alpha \leq -1$)					
FM3 $\alpha = 1$	100	5.0000	11	2.2096	7.47×10^{-07}
FM3 $\alpha = 0.5$	100	4.0927	12	5.15×10^{-01}	1.88×10^{-08}
Block 2: Convergence conditions are not satisfied					
FM3 $\alpha = -1$	100	3.9769	16	9.30×10^{-02}	7.88×10^{-11}
FM3 $\alpha = 2$	100	3.5161	10	1.1368	8.26×10^{-07}

$\|F(X_{k+1})\|_2 = \|I - AX_{k+1}\|_2 < 10^{-6}$. Tables 1, 2 and 3 show the results obtained by approximating the inverse of nonsingular random matrices of size n, where $n = 50$, $n = 100$ and $n = 500$, respectively. The initial estimate used for each method is $X_0 = \frac{A^T}{\| A \|_2^2}$, satisfying the convergence hypothesis of Theorem 2.1. Moreover, in each table, the selected values of α correspond to $\alpha = 1$ and $\alpha = 0.5$, where convergence conditions are met; also, values of $\alpha \notin [0, 1]$ are used, $\alpha = -1$ and $\alpha = 2$; in these cases we cannot ensure convergence.

To check the theoretical convergence order p, we use the approximate computational convergence order (COC), introduced by Jay (see [4]) and defined as:

$$p \approx COC = \frac{\ln \left(\|F(X_{k+1})\|_2 / \|F(X_k)\|_2 \right)}{\ln \left(\|F(X_k)\|_2 / \|F(X_{k-1})\|_2 \right)}.$$

Table 3: Results obtained by approximating the inverse of a random matrix of order $n = 500$.

Method	n	COC	it	$\| X_{k+1} - X_k \|_2$	$\| I - AX_k \|_2$
Block 1: Convergence conditions are met ($0 \le \alpha \le -1$)					
FM3 $\alpha = 1$	500	5.0000	18	51.649	1.73×10^{-08}
FM3 $\alpha = 0.5$	500	1.6447	20	2.70×10^{-02}	1.24×10^{-11}
Block 2: Convergence conditions are not satisfied					
FM3 $\alpha = -1$	500	3.9736	26	6.2734	2.68×10^{-10}
FM3 $\alpha = 2$	500	2.0119	17	8.19×10^{-02}	1.10×10^{-11}

4 Conclusions

In this manuscript, we have designed a parametric family of iterative methods without memory to approximate the inverse of a nonsingular complex matrix, analyzed their order of convergence, and performed numerical tests that confirm the theoretical results.

The numerical results show that the iterative methods corresponding to values of α inside the region of convergence work properly. However, for some parameter values that do not hold the convergence condition, results even better than the previous ones are obtained, meanwhile other converge more slowly.

Due to this behavior, we consider that a dynamical analysis must be performed, in order to detect those members of the class of iterative methods with good stability properties, with independence of the initial estimation and the belonging of the parameter to the convergence interval.

References

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