The hybrid time scales of an electromechanical system

Roberta Lima♭[1] Rubens Sampaio♮

(♭) PUC-Rio, Mechanical Engineering Department
Rua Marquês de São Vicente, 225, Gávea, Rio de Janeiro, RJ, Brazil, 22451-900.
(♮) PUC-Rio, Mechanical Engineering Department,
Rua Marquês de São Vicente, 225, Gávea, Rio de Janeiro, RJ, Brazil, 22451-900.

1 Introduction

Electromechanical systems are an interesting type of dynamical systems. They are composed by two interacting subsystems, a mechanical and an electromagnetic. To properly characterize the dynamics of an electromechanical system, it is not sufficient to characterize the dynamics of each subsystem independently. It is necessary to include in the mathematical model of the system dynamics the mutual influence between the two subsystems [1, 2, 3]. The state of an electromechanical system must involve mechanical and electromagnetic variables, as for example, positions, velocities, angles, currents, and charges [4, 5]. At least one mechanical and one electromagnetic variables should be considered in the dynamics parametrization. If variables of just one nature are used, the system can not be classified as electromechanical. It will be purely mechanical or purely electromagnetic. Thus, the smaller number of variables that should be used to parametrize the dynamics of an electromechanical system is two.

The fact that mechanical and electromagnetic variables must appear in the parametrization is reflected in the initial value problem (IVP) that gives the system dynamics. The initial value problem is composed by a set of differential equations and initial conditions with these two types of variables. In the set, the mutual interaction between the mechanical and an electromagnetic subsystems does not appear as a functional relation. The mutual interaction varies with the state of the subsystems and, consequently, depends on initial conditions. The dynamic behavior of an electromechanical system depends on this mutual interaction, i.e., the phenomena present in the system response reflect this interplay between the mechanical and electromagnetic subsystems. In this paper, we focus in a special phenomenon: oscillations. We analyze the oscillatory response of the simplest electromechanical system. The system is composed by a DC motor connected to a rigid disc, a motor-disc system. This system has the minimum number of elements necessary to be classified as an electromechanical system. It is a bare minimum to study oscillatory response of electromechanical systems and to make modal analysis. One of the reasons to address the problem in this bare minimum system is to highlight that the mutual interaction between the mechanical and electromagnetic subsystems provokes an oscillatory response. Besides, the system was chosen as simple as possible so that the analyses could be done analytically. Natural frequency and modes are computed. Differently from purely mechanical systems [6, 7], here these parameters involve

[1] robertalima@puc-rio.br

mechanical and electromagnetic variables, i.e., the computed natural frequency and modes are hybrid.

The computed hybrid natural frequency is the frequency at which the electromechanical system responds when there is no external excitation acting over it, that is, when the system is free. In our case, this means no external torque acting over the disc and no external source voltage applied over the electric circuit of the DC motor. In this situation, the hybrid natural frequency also represents the frequency at which occurs the interplay of energies between the mechanical and the electromagnetic subsystems. A consequence of the fact the hybrid natural frequency characterizes the free response of our electromechanical system is that the system response does not have an electromagnetic and a mechanical time scales. It has just one time scale, the one arising from this hybrid natural frequency. We believe that this result brings an important contribution to the area of electromechanical systems since several references in the literature affirm that the dynamics of an electromechanical system can be characterized by mechanical and electromagnetic time scales. Some examples where the mistakes appear are references 1, 2 and 3 of [3].

2 Dynamics of the electromechanical system

The electromechanical system analyzed in this paper is a DC motor connected to a disc as shown in Fig. 1.

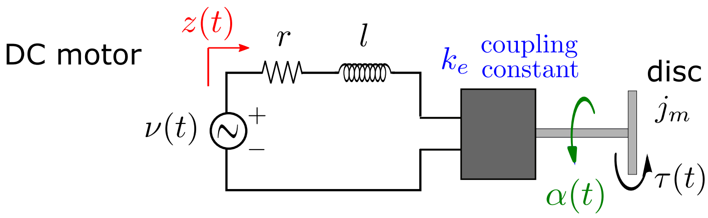

Figure 1: Electromechanical system.

The IVP that characterizes the system dynamics is defined as follows. Find $(\alpha,\ z)$ such that, for all $t > 0$,

$$\begin{aligned} l\ddot{z}(t) + r\,\dot{z}(t) + k_e\dot{\alpha}(t) &= \nu(t)\,, \\ j_m\ddot{\alpha}(t) + b_m\dot{\alpha}(t) - k_e\dot{z}(t) &= \tau(t)\,, \end{aligned} \tag{1}$$

with the initial conditions $\dot{\alpha}(0) = \theta_0$, $\alpha(0) = \alpha_0$, $\dot{z}(0) = c_0$ and $z(0) = z_0$. In these equations, t is the time, ν is the source voltage, z is the electric charge, $\dot{\alpha}$ is the angular speed of the disc, l is the electric inductance, j_m is the disc moment of inertia, b_m is the damping ratio in the transmission of the torque generated by the motor, k_e is the motor electromagnetic force constant, r is the electrical resistance, and τ is an external torque made over the disc. Writing Eq. (1) in matrix form, and assuming $b_m = 0$ and $r = 0$ to get a conservative system, we obtain:

$$\begin{bmatrix} l & 0 \\ 0 & j_m \end{bmatrix} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\alpha}(t) \end{bmatrix} + \begin{bmatrix} 0 & k_e \\ -k_e & 0 \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ \dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} \nu(t) \\ \tau(t) \end{bmatrix}, \tag{2}$$

$$M\ddot{\mathbf{Y}}(t) + G\dot{\mathbf{Y}}(t) = \mathbf{F}(t)\,, \tag{3}$$

where M and G will be called inertia and gyroscopic matrices respectively and $\mathbf{Y} = \begin{bmatrix} z \\ \alpha \end{bmatrix}$. The initial conditions become $\dot{\mathbf{Y}}(0) = \begin{bmatrix} c_0 \\ \theta_0 \end{bmatrix}$ and $\mathbf{Y}(0) = \begin{bmatrix} z_0 \\ \alpha_0 \end{bmatrix}$. Matrix G is skew symmetric, i.e., $G^T = -G$, where $\square^T$ indicates the transpose.

Making $\mathbf{X} = \dot{\mathbf{Y}}$ in Eq. (3), we propose as solution to the associated homogeneous equation $\mathbf{X}_h = \mathbf{U}\, e^{\lambda\, t}$, where $\mathbf{U}$ is a non-zero constant vector and λ a scalar. Substituting the proposed general solution into the the associated homogeneous equation, we get $(A - \lambda\, I)\mathbf{U} = \mathbf{0}$, which forms an eigenvalue problem. Since $\mathbf{U} \neq \mathbf{0}$, the matrix $(A - \lambda\, I)$ is singular. Thus:

$$\det\Big(A - \lambda\, I\Big) = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k_e^2}{l\, j_m} = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm\frac{k_e}{\sqrt{l\, j_m}}\, i\,, \tag{4}$$

where $i = \sqrt{-1}$. Substituting the two eigenvalues $\lambda_{1,2}$ into the eigenvalue problem, it is possible to write $(A - \lambda_1\, I)\mathbf{U}_1 = \mathbf{0}$ and $(A - \lambda_1\, I)\mathbf{U}_2 = \mathbf{0}$. For $\lambda_1 = \dfrac{k_e}{\sqrt{l\, j_m}}\, i$, the associated eigenvector is $\mathbf{U}_1 = \begin{bmatrix} i\, j_m/\sqrt{l\, j_m} \\ 1 \end{bmatrix}$. For $\lambda_2 = -\dfrac{k_e}{\sqrt{l\, j_m}}\, i$, the associated eigenvector is $\mathbf{U}_2 = \begin{bmatrix} -i\, j_m/\sqrt{l\, j_m} \\ 1 \end{bmatrix}$. The eigenvalues $\lambda_{1,2}$ give a natural frequency of the system $\omega_n = \dfrac{k_e}{\sqrt{l\, j_m}}$. The eigenvectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are modes. Observe that the natural frequency, ω_n, and the modes are hybrid. They involve mechanical and electromagnetic parameters. Since two pairs of eigenvalues and eigenvectors were found, the general solution of the associated homogeneous equation will be a linear combination of the two found solutions $e^{\lambda_1\, t}\, \mathbf{U}_1$ and $e^{\lambda_2\, t}\, \mathbf{U}_2$.

3 Energetic analysis

To make an energetic analysis of our electromechanical system, we start multiplying Eq. (3) on the left by $\dot{\mathbf{Y}}^T$. Considering a homogeneous system with $\mathbf{F} = \mathbf{0}$, we get

$$\begin{array}{rcl} \dot{\mathbf{Y}}^T(t)M\ddot{\mathbf{Y}}(t) + \dot{\mathbf{Y}}^T(t)G\dot{\mathbf{Y}}(t) & = & 0 \\ l\ddot{z}\dot{z} + j_m\ddot{\alpha}\dot{\alpha} & = & 0 \\ \dfrac{d}{dt}\left[\dfrac{1}{2}l\dot{z}^2(t) + \dfrac{1}{2}j_m\dot{\alpha}^2(t)\right] & = & 0\,. \end{array} \tag{5}$$

The term $\frac{1}{2}l\dot{z}^2$ represents the magnetic energy of the system and the term $\frac{1}{2}j_m\dot{\alpha}^2$ the kinetic energy. The system does not have potential energies, neither mechanical nor electromagnetic. Observing Eq. (5) it is possible to verify that the sum of the magnetic and kinetic energies is constant, that is, the analyzed homogeneous electromechanical system is conservative. The free response of our motor-disc system is characterized by the interplay of kinetic and magnetic energies. This energy interplay provokes an oscillatory response. Observe that what provokes free oscillatory response in purely mechanical systems is the interplay of kinetic and potential energies.

The total energy present in our homogeneous electromechanical system is defined by the initial conditions of current in the electric circuit of the motor, $c(0) = c_0$, and speed of the disc, $\dot{\alpha}(0) = \theta_0$. Thus:

$$\frac{1}{2}l\dot{z}^2(t) + \frac{1}{2}j_m\dot{\alpha}^2(t) \;=\; \frac{1}{2}lc_0^2 + \frac{1}{2}j_m\theta_0^2\,. \tag{6}$$

The free response of the system is characterized by an interplay of kinetic and magnetic energies. The phase diagram of $\dot{\alpha}$ and c is a center around the point (0, 0).

4 Conclusions

In this paper, the oscillatory response of the simplest electromechanical system is analyzed. The system dynamics is written in terms of mass and gyroscope matrices. The term gyroscope is employed only in the sense of an antisymmetric matrix, which couples two subsystems, one mechanical, another electromagnetic. The other matrix that appears, the mass matrix, is a sort of inertia term, but with two different inertia, mechanical and electromagnetic. The deal show the interplay between this two terms results in a very interesting dynamics, somehow similar to the harmonic oscillator. After doing the work, it appears quite simple, but one should also ask why it was not done before. Natural frequency and modes were computed for the electromechanical system. Since the computed parameters involve mechanical and electromagnetic variables, they are hybrid. The hybrid natural frequency is the frequency at which occurs the interplay of energies between the mechanical and the electromagnetic subsystems. The hybrid modes forms a basis of a vector space that can be used to represent and decouple the system dynamics. It should be remarked that the response of the simplest electromechanical system does not have an electromechanical and a mechanical time scales. It has just one time scale, the one arising from the hybrid natural frequency.

Acknowledgments

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