Scattering of kinks with a restriction on the sphere

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1 Introduction

Topological defects have been successfully recognized over the last fifty years as the source of a wide variety of non-linear phenomena arising in different physical frameworks. In general, topological defects are solutions in field theory models which cannot decay to the homogeneous ground state, the vacuum.

Some physical applications require the use of non-linear Sigma field theory models with different target spaces. This is, for example, the scenario found in spintronics, where the fields in the effective theory describe the continuous limit of spin chains in magnetic materials. In this framework the target manifold is the sphere $\mathbb{S}^2$, that is, a three-component scalar field must comply with the constraint $\phi_1^2+\phi_2^2+\phi_3^2 = R^2$. For example, Haldane constructed a $O(3)$ non-linear sigma field theory model to describe the low-energy dynamics of large-spin one-dimensional Heisenberg antiferromagnets, see [5].

2 A non-linear $\mathbb{S}^2$-Sigma model

In this work we will study another non-linear $\mathbb{S}^2$-Sigma model in $(1+1)$ dimensional Minkowski space-time. The dynamics of the system will be governed by the functional action:

$$S[\phi] = \int dt \int dx \left\{ \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 + \partial_\mu\phi_3\partial^\mu\phi_3\right) - V\left(\phi_1, \phi_2\phi_3\right) \right\}, \tag{1}$$

where ϕ_i with $i = 1, 2, 3$ are real scalar fields in the sphere, that is, $\phi_a(t,x) \in \text{Maps}\left(\mathbb{R}^{1,1}, \mathbb{S}^2\right)$. The Minkowski metric is chosen as $\eta_{\mu\nu} = \text{diag}\,(1, -1)$ and Einstein summation convention is only applied to space-time indexes.

Since, the scalar fields are constrained in a sphere of radius R, then the following equation must be also satisfied:

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = R^2. \tag{2}$$

The potential term $V(\phi_1, \phi_2, \phi_3)$ considered in (1) can be expressed as:

$$V\left(\phi_1, \phi_2, \phi_3\right) = \frac{1}{2}\left(\alpha_1^2\phi_1^2 + \alpha_2^2\phi_2^2 + \alpha_3^2\phi_3^2\right) + \frac{\beta}{2}\frac{\phi_1^2\phi_2^2}{\phi_1^2+\phi_2^2} + C. \tag{3}$$

In order for this potential to be definitively semi-positive, an appropriate C will be chosen later, and with no loss of generality we consider $\alpha_1^2 > \alpha_2^2 > \alpha_3^2 \geq 0$.

This potential has a singular point where $\phi_1 = \phi_2 = 0$, but given that $\lim_{(\phi_1,\phi_2)\to(0,0)}\left(\frac{\phi_1^2\phi_2^2}{\phi_1^2+\phi_2^2}\right) \to 0$, therefore the potential is well-defined in that singular point.

Minimizing (1), the fields equations can be obtained as:

$$
\begin{aligned}
&\partial_\mu^2\phi_1+\alpha_1^2\phi_1+\beta\frac{\phi_1\phi_2^4}{\left(\phi_1^2+\phi_2^2\right)^2}=0,\\
&\partial_\mu^2\phi_2+\alpha_2^2\phi_2+\beta\frac{\phi_2\phi_1^4}{\left(\phi_1^2+\phi_2^2\right)^2}=0,\\
&\partial_\mu^2\phi_3+\alpha_3^2\phi_3=0,
\end{aligned}
\tag{4}
$$

in addition to the relation (2).

The functional action (1) is Lorentz invariant. Therefore, given a finite energy static solution $\phi_1(x)$, $\phi_2(x)$ $\phi_3(x)$ the model admits solutions of type

$$
\phi_i\,(t,x)=\phi_i\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\quad,\quad i=1,2,3, \tag{5}
$$

for any velocity v with a magnitude lower than the velocity of light which is chosen as $c=1$. Those solutions are therefore known as solitary waves.

Using (2) we can obtain ϕ_3 in function of the fields ϕ_1 and ϕ_2 and substitute it in (1). In this case, the functional actions takes the form:

$$
\begin{aligned}
S[\phi]=\int dt\int dx\,\Bigg\{&\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1+\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2+\\
&+\frac{1}{2}\frac{\left(\phi_1\partial_\mu\phi_1+\phi_2\partial_\mu\phi_2\right)\left(\phi_1\partial^\mu\phi_1+\phi_2\partial^\mu\phi_2\right)}{R^2-\phi_1^2-\phi_2^2}-V_{\mathbb{S}^2}(\phi_1,\phi_2)\Bigg\}\ ,
\end{aligned}
\tag{6}
$$

where now the potential energy term is

$$
V_{\mathbb{S}^2}\left(\phi_1,\phi_2\right)=\frac{\alpha_1^2-\alpha_3^2}{2}\left(\phi_1^2+\frac{\alpha_2^2-\alpha_3^2}{\alpha_1^2-\alpha_3^2}\phi_2^2+\frac{\beta}{\alpha_1^2-\alpha_3^2}\frac{\phi_1^2\phi_2^2}{\phi_1^2+\phi_2^2}\right)+C+\frac{\alpha_3^2}{2}R^2. \tag{7}
$$

In order to $V_{\mathbb{S}^2}\left(\phi_1,\phi_2\right)$ to be positive semi-definite is necessary that $C=\frac{-R^2\alpha_3^2}{2}$. And the vacua of the model, which corresponds to the set of solutions $\mathcal{M}=\{\phi_v\in\mathbb{R}^3 : V\left(\phi_v\right)=0\}$, are:

$$
\mathcal{M}=\{v_N=(0,0,R),\ v_S=(0,0,-R\}.
$$

Using the following change of variables

$$
x^\mu\to\frac{x^\mu}{\sqrt{\alpha_1^2-\alpha_3^2}}\quad,\quad\sigma^2=\frac{\alpha_2^2-\alpha_3^2}{\alpha_1^2-\alpha_3^3}\quad,\quad\gamma=\frac{\beta}{4\left(\alpha_1^2-\alpha_3^2\right)} \tag{8}
$$

it is possible to rewrite the potential as:

$$
V_{\mathbb{S}^2}=\frac{1}{2}\left(\phi_1^2+\sigma^2\phi_2^2+\frac{4\gamma\phi_1^2\phi_2^2}{\phi_1^2+\phi_2^2}\right)\ . \tag{9}
$$

Minimizing (6), one obtains the equations of motion with the constrain imposed:

$$
\partial_\mu^2\phi_1+\phi_1\left(\frac{\left(\phi_1\partial_\mu\phi_1+\phi_2\partial_\mu\phi_2\right)^2}{\left(R^2-\phi_1^2-\phi_2^2\right)^2}+\frac{\left(\partial_\mu\phi_1\right)^2+\phi_1\partial_\mu^2\phi_1+\phi_2\partial_\mu^2\phi_2+\left(\partial_\mu\phi_2\right)^2}{R^2-\phi_1^2-\phi_1^2}\right)-\frac{\partial V_{\mathbb{S}^2}}{\partial\phi_1}=0, \tag{10}
$$

$$
\partial_\mu^2\phi_2+\phi_2\left(\frac{\left(\phi_1\partial_\mu\phi_1+\phi_2\partial_\mu\phi_2\right)^2}{\left(R^2-\phi_1^2-\phi_2^2\right)^2}+\frac{\left(\partial_\mu\phi_1\right)^2+\phi_1\partial_\mu^2\phi_1+\phi_2\partial_\mu^2\phi_2+\left(\partial_\mu\phi_2\right)^2}{R^2-\phi_1^2-\phi_1^2}\right)-\frac{\partial V_{\mathbb{S}^2}}{\partial\phi_2}=0. \tag{11}
$$

And the energy functional associated to the action (6) is

$$
\begin{aligned}
E=\int dx\Bigg\{&\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2+\\
+&\frac{\left(\phi_1\partial_\mu\phi_1+\phi_2\partial_\mu\phi_2\right)\left(\phi_1\partial^\mu\phi_1+\phi_2\partial^\mu\phi_2\right)}{R^2-\phi_1^2-\phi_2^2}+V_{\mathbb{S}^2}\left(\phi_1,\phi_2\right)\Bigg\}\ .
\end{aligned}
\tag{12}
$$

In order to the solutions of the system have finite energy associated, the following boundary conditions must be satisfied:

$$\lim_{x\to\pm\infty} \frac{d\phi_i}{dx} = 0 \ , \quad \lim_{x\to\pm\infty} \phi_i = 0 \ . \tag{13}$$

Additionally, spherical coordinates can be employed in this problem, in this way, the potential term in spherical coordinates becomes:

$$V(\theta,\varphi) = \frac{R^2}{4}\sin^2\theta\left(1+\gamma+\sigma^2+\overline{\sigma}^2\cos(2\varphi)-\gamma\cos(4\varphi)\right) \ , \tag{14}$$

where $\bar{\sigma}$ is defined as $\overline{\sigma}^2 = 1-\sigma^2$.

And using spherical coordinates in (1) the action can also be express as,

$$S[\theta,\varphi] = \int dx \int dt \left\{ \frac{1}{2}R^2\left(\partial_\mu\theta\partial^\mu\theta + \sin^2\theta\partial_\mu\varphi\partial^\mu\varphi\right) - V(\theta,\varphi)\right\} . \tag{15}$$

and the field equations obtained minimizing the previous action are

$$R^2\partial_\mu\partial^\mu\theta + \frac{R^2}{4}\left(1+\gamma+\sigma^2+\overline{\sigma}^2\cos(2\theta)-\gamma\cos(4\varphi)-2\left(\partial_\mu\varphi\right)^2\right)\sin(2\theta) = 0, \tag{16a}$$

$$R^2\partial_\mu\left(\sin^2\theta\partial^\mu\varphi\right) - \frac{R^2}{2}\sin^2\theta\left(1-\sigma^2-4\gamma\cos(2\varphi)\right)\sin(2\varphi) = 0 \ . \tag{16b}$$

Solitary waves (5) can be also obtain in spherical coordinates and have this form:

$$\theta(t,x) = \theta\left(\frac{x-vt}{\sqrt{1-v^2}}\right) \ , \quad \varphi(t,x) = \varphi\left(\frac{x-vt}{\sqrt{1-v^2}}\right) \ . \tag{17}$$

Thus, the functional energy (12) in spherical coordinates is as follows:

$$E = \int dx \left\{ \frac{1}{2}R^2\left(\partial_\mu\theta\partial^\mu\theta + \sin^2\theta\partial_\mu\varphi\partial^\mu\varphi\right) + V(\theta,\varphi)\right\} . \tag{18}$$

3 Topological kinks

Without loss of generality, let us consider $R = 1$ in (16b). This equation is satisfied for constant values of the azimuthal spherical coordinate φ. Depending on which constant values φ we choose, the equation (16a) can take one of the following forms:

$\mathbf{K_1/K_1^*}$ Kinks: On the orbits $\varphi = 0$ and $\varphi = \pi$, the expression (16a) takes the form:

$$\partial_\mu\partial^\mu\theta + \frac{1}{2}\sin 2\theta = 0 \ , \tag{19}$$

where the the static solution obtained are static Kinks type solutions:

$$\theta_{K_1}(x) = 2\arctan e^{\pm(x-x_0)} \ . \tag{20}$$

Making use of (18) the energy calculated is $E = 2$.

$\mathbf{K_2/K_2^*}$ Kinks: In the half meridians $\varphi = \frac{\pi}{2}$ and $\varphi = \frac{3\pi}{2}$, the equation of motion (16a) can be expressed as:

$$\partial_\mu\partial^\mu\theta + \frac{\sigma^2}{2}\sin 2\theta = 0 \ , \tag{21}$$

which are also static Kinks solutions:

$$\theta_{K_2}(x) = 2\arctan e^{\pm\sigma(x-x_0)} \ . \tag{22}$$

Using (18) the energy obtained is $E = 2\sigma$.

$\mathbf{K_{a^\pm}/K^*_{a^\pm}}$ **Kinks:** In the case of those orbits, $\varphi = \frac{1}{2}\arctan\frac{\pm\sqrt{16\gamma^2-(1-\sigma^2)^2}}{1-\sigma^2}$, for $1-\sigma^2 < 4\gamma$, the equation of motion (16a) obtained is:

$$\partial_\mu \partial^\mu \theta + \frac{\lambda}{2}\sin 2\theta = 0 \ , \tag{23}$$

where $\lambda = \frac{(4\gamma+(1+\sigma)^2)(4\gamma+(1-\sigma)^2)}{16\gamma}$ and the static solution calculated is

$$\theta_{K_{a^\pm}}(x) = \arctan e^{\pm\lambda(x-x_0)} \ . \tag{24}$$

All these solutions are represented in Fig. 1 where two figures are shown. In both of them the three type of solutions are given to illustrate their stability.

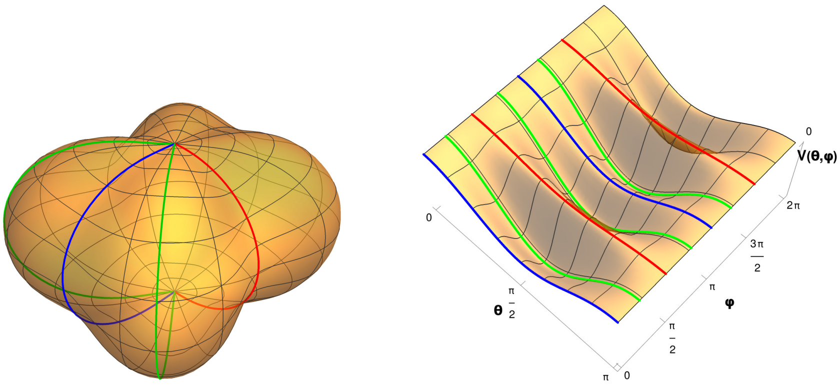

Figure 1: The potential $V(\theta,\varphi)$ is represented in both figures. In the left graphic, the distance between surface represented in yellow and the sphere of radius one with the same origin represents the value of $V(\theta,\varphi)$ in each point of the internal space. In the right graphic, $V(\theta,\varphi)$ is illustrated in function on the spherical coordinates θ and φ of the internal space. The solutions $\mathbf{K_1/K_1^*}$ (Blue), $\mathbf{K_2/K_2^*}$ (Red) and $\mathbf{K_{a^\pm}/K^*_{a^\pm}}$ (Green) are illustrated in both figures.

4 Scattering. Numerical solution of Differential Algebraic Equations

In the MME&HB 2024 International Conference we would like to focus our investigation on the dynamics of different topological Kinks of the model described in the previous section.

To achieve this purpose, we will solve numerically the equations of motion (4) in Cartesian coordinates with the constrain (2). After spatial discretizations in the PDEs, they become systems of differential algebraic equations (DAEs) because of the constrain.

The functional action of the theory (1) is invariant under Lorentz transformations (5). It allows us to give a initial velocity v to the topological Kinks mentioned in the previous section.

As initial conditions we consider a Kink solution and an anti-Kink solution well separated. We choose the Kink solution will be in the region $x < 0$ with an initial position $x_{01} = -10$ and an anti-Kink solution in the region $x > 0$ with an initial position $x_{02} = 10$. Without loss of generality, both will have the same magnitude of the initial velocity but with opposite directions. The value of the parameters of the model elected where $\gamma = 1$ and $\sigma = 1.5$. We shall apply absorbing fourth order boundary conditions [2] so that radiation does not bounce back in the border of the region computation.

In this case, the index of the field equations in cartesian coordinates (4) together with the constraint (2) is 3, and therefore an index reduction is necessary to obtain DAEs with index 1. Later the Forest-Ruth's fourth-order symplectic method based on Hamilton equations [1] is employed, combined with the projection technique described in [4, §VII] or [3, §VII]. The scattering will be performed for different values of initial velocity and a diagram of velocities will be built in each case with a velocity step size $\Delta v = 0.01$.

References

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