Extension of certain temporal transformations of the elliptic two-body problem to hyperbolic motion

José Antonio López Ortí[♭,1] , María José Martínez Usó and Francisco José Marco Castillo[♭]

(♭) Departamento de Matemáticas, Universitat Jaume I de Castellón. IMAC.
Av. Sos Baynat s/n, Castellón, Spain.

(♮) Departamento de Mateática Aplicada, Universitat Politècnica de Valencia. IUMPA.
Camino de Vera s/n, Valencia, Spain.

1 Introduction

One of the main challenges of celestial mechanics is the study of the well-known two-body problem, which serves as a basis for studying more complex situations, such as planetary motion in the solar system, which can be considered a general case of the problem of two disturbed bodies,
The basic problem is also very useful for studying the efficiency of integration algorithms because the analytical solution is very well determined [1], [9], , [12], [13], [21]. The elliptical case is an example, when the eccentricity is high, since numerous difficulties appear. The values provided by numerical integration with the exact ones can be compared to determine the goodness of the calculation. Another difficulty in which the comparison is of great help is when changes occur in the type of movement, going from elliptical to parabolic and hyperbolic and vice versa, due to the influence of the disturbances of the different celestial bodies between them.
The equation of relative motion of the secondary with respect to the primary is given by:

$$\frac{d^2\vec{r}}{dt^2} = -\mu\frac{\vec{r}}{r^3}.\vec{F}$$

where $\vec{r}$ is the radius vector of the secondary with respect to the primary, $\mu = G(m_1 + m_2)$, G is the gravitational constant, m_1, m_2 the masses of the primary and of the secondary respectively and $\vec{F}$ the perturbative forces, which in general are small and in this work will be considered null.

The performance of the numerical methods is, in general, good. However, in the elliptical case, when the eccentricity is high, difficulties usually arise due mainly to the fact that natural time is not a variable in accordance with the dynamics of the problem since, due to Kepler's second law, at equal time intervals, the concentration of base points for integration is much lower in the periapsis region than in the apoapsis region. The concentration of points is also desirable to be more significant in the regions with greater curvature.

There are various alternatives to solve this problem. In this article, we will use the technique known as analytical regularization of the integration step, also called analytical reparametrization, which will provide a distribution of points on the orbit that is more appropriate to the dynamics of the problem. This method can be combined with variable step methods with any class of integrators,

[1]lopez@mat.uji.es

including simplex integrators and other techniques. An interesting review of reparametrization methods and their applications to various problems can be found in [4].

In the case of elliptical motion, numerous reparametrizations have been studied. In 1912, Sundman introduced a change in the time variable through the transformation $r\,d\tau = dt$, known as the Sundman transformation [20]. In the case of elliptical motion, this transformation produces the eccentric anomaly g for which the normalization factors $r/a\,dg = ndt$ are introduced, where n represents the mean motion. Note that $ndt = dM$ where M is the mean anomaly.

In the case of hyperbolic motion, this transformation leads to the hyperbolic variable H.

Numerous authors have used this method to obtain temporal reparameterizations Ψ in the form $dM = Q(r)d\Psi$ where $Q(r)$ is the so-called partition function $\Psi = \Psi(M)$ is a periodic function of the mean anomaly M of period 2π which satisfies $\Psi(0) = 0$, $\Psi(\pi) = \pi$. Thus, Nacozy introduced a new parameter τ, described by $r^{\frac{3}{2}}d\tau = dt$ [19]. Janin and Bond extended this transformation as Ψ_α defined by $r^\alpha d\Psi_\alpha = dt$ [10], [11], which are known as generalized Sudman anomalies, Brumberg [2] introduced the regularized arc length s^* by $vds^* = dt$, where v is the velocity of the secondary. Brumberg and Fukushima introduced the elliptical anomaly ω as $\omega = \frac{\pi u}{2K(e)} - \frac{\pi}{2}$, where $\operatorname{am} u = g + \frac{\pi}{2}$ [3].

All the above variables can be reduced to anomalies by introducing a normalization factor appropriate to the interval $[0.2\pi]$ for one revolution. The classic mean M, eccentric g, and mean f anomaly are temporal variables included in the family of generalized Sudman anomalies. López [17] defines the semifocalanomaly Ψ as the mean between f and f', but not the reduced arc length nor the elliptical anomaly.

To solve this difficulty, López introduced [16] the biparametric family of anomalies $\Psi_{\alpha,\beta}$ defined as:

$$_{\alpha,\beta}r^\alpha r'^\beta d\Psi_{\alpha,\beta} = dM \tag{1}$$

so that all the anomalies defined above can be found in it. Also included in the biparametric family are the antifocal anomaly f', defined by Fukushima [8], and the semifocal anomaly Ψ defined by López [17], [18] as the half-sum of f and f'.

The semifocal anomaly does not include all anomalies since there are interesting reparametrizations, [7], [5], [14], [15] not included in this family, which does include the most used ones.

The next step is to extend the [16] transformation to hyperbolic motion to extend the study carried out in [18] to this case.

This paper is focused on studying a new geometric point of view about the biparamétric family in the hyperbolic motion case. This section presents the background and the primary goal of this paper. In Section 2, we study the relation between the curvature and the vector radii r r', and from them, the biparametric family of anomalies is related through the vector radius r and the curvature. In section 3, the main conclusions of this study will be discussed.

2 The biparametric family of anomalies as a function of the vector radius and the curvature in the case of hyperbolic motion.

In this section, the curvature in the hyperbolic motion is determined as a function of the eccentricity e and the hyperbolic variable F. For this variable, it is well known that:

$$\xi = a(e - coshH), \quad \eta = a\sqrt{e^2 - 1}\sinh H \tag{2}$$

where (ξ, η) are the orbital coordinates referred to the primary focus placed in the point F, $F\xi$ running to the periapsis region and $F\eta$ making a direct orthogonal system with $F\xi$ the motion of the secondary whit respect to the primary, direct in the system (F, ξ, η). On the other hand, the vector radius r of the secondary with respect to primary and the vector radius r' with secondary F' are given by:

$$r = a(e\cosh H - 1), \quad r' = a(e\cosh H + 1). \tag{3}$$

The curvature $\kappa(t)$ of a planar parametric curve $\vec{r}(t) = (x(t), y(t))$ is given by [6]

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} \tag{4}$$

Applying this formula to the equation of the ellipse parametrized by the hyperbolic variable F $\vec{r} = (a(ecoshFH - 1), a\sqrt{e^2-1}\sinh H)$ we obtain:

$$\kappa(H) = \frac{\sqrt{e^2-1}}{(a\sqrt{e^2\cosh^2 H - 1})^3}. \tag{5}$$

On the other hand, we have that the product of vector radii r and r' is given by:

$$r\,r' = a^2(e^2\cosh^2 H - 1), \tag{6}$$

and comparing with (5) we obtain:

$$(r\,r')^3\kappa(H)^2 = a^2(e^2-1), \tag{7}$$

and so (r r') can be represented using the curvature as

$$r\,r' = a\,a^{2/3}\sqrt[3]{e^2-1}\kappa(H)^{-2/3}. \tag{8}$$

For this reason the biparametric family of anomalies $\Psi_{\alpha,\beta}$ can be rewritten as $\Phi_{\gamma,\delta}$ where:

$$C_{\gamma,\delta}r^{\gamma}\kappa(g)^{\delta}d\Psi_{\gamma,\delta} = dM \tag{9}$$

where $\gamma = \alpha - \beta$, $\delta = -\frac{2}{3}\beta$, and $C_{\gamma,\delta} = K_{\gamma,\delta}a^{2/3}\sqrt[3]{e^2-1}$

3 Conclusions

This paper provides a new point of view on the biparametric family of anomalies.

This family can be rewritten int the form $\Phi_{\gamma,\delta}$ where $C_{\gamma,\delta}r^{\gamma}\kappa(g)^{\delta}d\Psi_{\gamma,\delta} = dM$. These anomalies depend on two factors r^{γ} and $\kappa(g)^{\delta}$, which allows a simple interpretation of the reparametrization. Taking n points on the ellipse with anomaly $\Phi_{\gamma,\delta} = k\,h$ $h = 2\pi/n$, $k = 1,\ldots,n$ the mean of the first factor is a displacement of the points from the region of apoapsis to the periapsis one. The second factor $\kappa(g)^{\delta}$ represents the dependence of these transformations on the curvature, which allows a distribution of integration base points that are not only dependent on the distance to the focus. However, this fact does not have such a clear interpretation as in the case of elliptical motion.

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