Block preconditioners for double saddle point linear systems arising in coupled poromechanics

L. Bergamaschi[♭,1], M. Ferronato[♭] and A. Martínez[♮]

(♭) Department of Civil Environmental and Architectural Engineering, University of Padua, Italy.

(♮) Department of Mathematics, Informatics and Geosciences, University of Trieste, Italy,

1 Introduction

Given positive integer dimensions n, m and p with $n \geq \max\{m, p\}$, consider the $(n+m+p) \times (n+m+p)$ double saddle point linear system

$$\mathcal{A}w \equiv \begin{bmatrix} A & B^T & 0 \\ B & -D & C^T \\ 0 & C & E \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite (SPD) matrix, $B \in \mathbb{R}^{m\times n}$ has full row rank, $C \in \mathbb{R}^{p\times m}$ has full rank, $D \in \mathbb{R}^{m\times m}$ and $E \in \mathbb{R}^{p\times p}$ are square positive definite matrices. Moreover $f \in \mathbb{R}^n$, $g \in \mathbb{R}^m$ and $h \in \mathbb{R}^p$ are given vectors. Such linear systems arise in a number of scientific applications including constrained least squares problems, constrained quadratic programming, magma-mantle dynamics, to mention a few. Similar block structures arise e.g. in liquid crystal director modeling or in the coupled Stokes-Darcy problem, and the preconditioning of such linear systems has been considered in Beik and Benzi (2022), Bakrani Balani et al. (2023). Due to the symmetry of the linear system, SPD preconditioners have attracted the attention of some authors, to be used in the framework of the MINRES method. See e.g. the analysis of block diagonal preconditioners in Bradley and Greif (2023); Sogn and Zulehner (2018), and of another SPD preconditioner, in the framework of multiple saddle-point symmetric linear systems, in Pearson and Potschka (2023); Bergamaschi et al. (2024b).

2 Two inexact block preconditioners

The aim of this contribution is to consider the inexact variants of two block preconditioners, and give bounds on the eigenvalues of the corresponding preconditioned matrices, extending the results provided in Bakrani Balani et al. (2024), which addresses the simpler case where $D \equiv 0$ and $E \equiv 0$.

We define

$$S = D + BA^{-1}B^T, \; X = E + CS^{-1}C^T$$

and $\widehat{A}, \widehat{S}$ and $\widehat{X}$ as symmetric positive definite approximations of A, S, and X, respectively.

[1] luca.bergamaschi@unipd.it

We will analyze the eigenvalue distribution of the preconditioned matrices $\mathcal{A}{\mathcal{P}_T}^{-1}$ and $\mathcal{A}{\mathcal{P}_D}^{-1}$, where

$$\mathcal{P}_T = \begin{bmatrix} \widehat{A} & B^T & 0 \\ 0 & -\widehat{S} & C^T \\ 0 & 0 & \widehat{X} \end{bmatrix}, \qquad \mathcal{P}_D = \begin{bmatrix} \widehat{A} & 0 & 0 \\ 0 & \widehat{S} & 0 \\ 0 & 0 & \widehat{X} \end{bmatrix}, \tag{2}$$

The relevant spectral properties of the preconditioned matrices will be given in terms of the eigenvalues of $\overline{A} = \widehat{A}^{-1}A, \overline{S} = \widehat{S}^{-1}\widetilde{S}$ and $\overline{X} = \widehat{X}^{-1}\widetilde{X}$ where

$$\begin{aligned} \widetilde{S} &= D + B\widehat{A}^{-1}B^T, \qquad \widehat{S} \approx \widetilde{S} \\ \widetilde{X} &= E + C\widehat{S}^{-1}C^T, \qquad \widehat{X} \approx \widetilde{X} \end{aligned}$$

Our spectral analysis, which is conducted by repeatedly using the main results proved in Bergamaschi (2012) shows that all the truly complex eigenvalues of $\mathcal{A}{\mathcal{P}_T}^{-1}$ are inside a circle with center 1 and radius 1, while we characterize the remaining real and positive eigenvalues of the same matrix, as well as the real eigenvalues of $\mathcal{A}{\mathcal{P}_D}^{-1}$, as roots of third order polynomials whose coefficients are suitable Rayleigh Quotients of $\overline{A}, \overline{S}$ and $\overline{X}$.

3 Preliminary numerical results

We now concentrate on a realistic problem taken from Frigo et al. (2022), in which a mixed hybrid finite element formulation for coupled poromechanics is considered. The double saddle-point coefficient matrix takes on the form

$$A_H = \begin{bmatrix} A_{uu} & A_{up} & 0 \\ A_{pu} & \widetilde{A}_{pp} + A_{\text{stab}} & \Delta t A_{p\pi} \\ 0 & A_{\pi p} & A_{\pi\pi} \end{bmatrix}, \tag{3}$$

where A_{stab} is a matrix accounting for the discretization of a stabilization term, and subscripts u, p, and π stands for the discrete displacement, velocity and pressure unknowns. Double subscripts in matrices account for the coupling between two of these variables. Details on the model, discretization and stabilization procedures can be found in Frigo et al. (2022). Matrix A_H in (3) is not symmetric, since $A_{up} = -A_{pu}$ and owing to the presence of Δt. It can be, however readily symmetrized by changing sign to the second block row, and multiplying the third by $-\Delta t$. This done, solving a system with A_H is equivalent to solving a system (with suitably modified right hand side) with

$$\mathcal{A} = \begin{bmatrix} A & B^T & 0 \\ B & -D & C^T \\ 0 & C & E \end{bmatrix}$$

as in (1) where

$$A \equiv A_{uu},\ B \equiv -A_{pu},\ C \equiv -\Delta T A_{\pi p},\ D = \widetilde{A}_{pp} + A_{\text{stab}},\ \text{and } E = -\Delta T A_{\pi\pi}.$$

3.1 Bounds validation on a small test case

The sizes of the blocks are $n = 3362,\ m = 1600, p = 3200$. To set up the preconditioner, we used as the preconditioner for A the algebraic multigrid provided by the Matlab function `hsl_mi20` while the following approximations for S and X have been set, following the suggestion in Frigo et al. (2022):

$$\widehat{S} = A_{pp} + \text{diag}\left(A_{\text{stab}} - A_{pu}\text{diag}(A_{uu})^{-1}A_{up}\right), \qquad \widetilde{X} = A_{\pi\pi} - \Delta t A_{\pi p}\widehat{S}^{-1}A_{p\pi}.$$

Recalling that A_{pp} is diagonal the whole first level Schur complement approximation $\widetilde{S}$ is also diagonal. This allows to construct exactly matrix $\widetilde{X}$. Then we further approximate $\widetilde{X}$ by

$$\widehat{X} = \mathrm{diag}(\widetilde{X}),$$

in view of the weak diagonal dominance of $\widetilde{X}$. In this section we use a right hand side corresponding to a random solution, and the exit test on the relative residual is used it a tolerance of 10^{-12}.

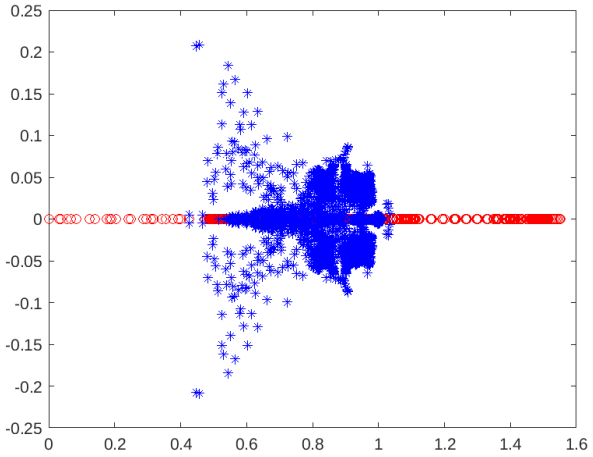

Figure 1: Eigenvalue distribution of the preconditioned matrix $\mathcal{A}\mathcal{P}^{-1}$ for the small size test case. Blue asterisks are the truly complex eigenvalues, red circles the real ones.

The eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ are shown in Figure 3.1. Our theoretical results yield that the complex eigenvalues are enclosed in a circle with center 1 and radius $\rho_{\min} = 3.8 \times 10^{-3}$. Regarding real eigenvalues, we have the following bounds, and *true* spectral intervals, for the preconditioned matrices with $\mathcal{P}_T$ and $\mathcal{P}_D$.

Triangular preconditioner: bounds			5.01e-05	4.02
Triangular preconditioner: true eigenvalues			5.01e-05	1.55
Diagonal preconditioner: bounds	-1.727	-0.3542	5.0098e-05	3.5151
Diagonal preconditioner: true eigenvalues	-1.125	-0.5546	5.0105e-05	0.9994

The triangular preconditioner $\mathcal{P}_T$ is used within the GMRES method, with a restart parameter equal to 50, while the SPD preconditioner $\mathcal{P}_D$ is employed to accelerate the MINRES Krylov subspace solver, with the following outcomes:

Table 1: Comparison between triangular and diagonal preconditioners.

solver(prec)	its	CPU	solver(prec)	its	CPU
GMRES ($\mathcal{P}_T$)	70	1.379	MINRES ($\mathcal{P}_D$)	185	3.266

3.2 Discussion

These preliminary results show that the developed bounds are very tight and well describe the behavior of the preconditioned iterative solvers. Moreover, and this is not completely obvious, despite the pleasant property of the MINRES method (short recurrence), it seems that the combination restarted-GMRES with triangular preconditioner is to preferred to the combination MINRES with diagonal preconditioner.

4 Ongoing work

Both the block triangular and diagonal preconditioners are being tried within the GMRES and MINRES Krylov solvers, respectively, onto large size test cases with a number of unknowns of order 10^6 arising from both Mixed Finite Element and Mixed Hybrid Finite Element discretizations of poro-elasticity groundwater problems (see Bergamaschi et al. (2024a)).

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