Random Walks meet M-matrices

Á. Carmona[♭], A. M. Encinas[♭], M.J. Jiménez[♭,1] and À. Martín[♭]

(♭) Departament de Matemàtiques, Universitat Politècnica de Catalunya
Carrer de Jordi Girona 31, Barcelona, Spain.

1 Introduction

Random walks are a very good way to represent systems with elements interacting with each other. This interaction can be modeled by a certain graph, assigning to each edge a certain transition probability. So, the movement through the graph can be represented by a random walk such that if we start in one of their nodes, we move randomly to one of its neighbors. Thus, we can talk about random walks on graphs [7].

The *transition matrix* P provides the probabilities for transitions between states in a random walk. Hence, it is possible to study its long-term behavior and its short-term behavior. It is very well known that for the first of them we have to consider an eigenvector of P, denoted by $\boldsymbol{\pi}$ and called *stationary distribution.* For the second one, it is defined the *mean first passage time* (MFPT) from state i to state j, that is, the expected number of steps to reach j from an initial state i. Moreover, the expected number of steps to randomly get any state i according to $\boldsymbol{\pi}$ is a constant independent of the initial state. The value is known as *Kemeny's constant.* We refer the reader to [6] for a detailed study of these concepts. There is a wide literature on the study of Kemeny's constant and the MFPT that relates these parameters to generalized inverses of the matrix $\mathsf{I}-\mathsf{P}$, see for instance [4, 5].

We aim at describing the MFPT and the Kemeny's constant using the generalized inverses associated with the combinatorial Laplacian instead of 1-inverses of the matrix $\mathsf{I}-\mathsf{P}$, since the combinatorial Laplacian is a symmetric and positive semidefinite Z-matrix, that is an M-matrix, and hence we can take advantage of its properties, see [1].

2 Random walks using the combinatorial Laplacian

Let $\Gamma=(V,E,c)$ be a network; that is, a finite and connected graph without loops nor multiples edges, with vertex or state set V and edges set E, being $c\in\mathbb{R}^+$ a conductance assigned to every edge. The cardinals of these two subsets are $|V|=n$ and $|E|=m$, respectively. We say that x is adjacent to y, $x\sim y$, if $\{x,y\}\in E$ and, in this case, we assign a *conductance* $c(x,y)>0$. Therefore, the conductance is a function $c:V\times V\longrightarrow[0,+\infty)$, such that $c(x,y)=c(y,x)$, and $c(x,y)=0$ if $\{x,y\}\notin E$. If we denote $\mathcal{C}(V)$ as the set of real functions on V, then $k\in\mathcal{C}(V)$ defined as $k(x)=\displaystyle\sum_{y\in V}c(x,y)$, is the *degree* of x. When $c(x,y)=1$ for any $x\sim y$, Γ is called *graph.* In this case, $k(x)$ is the number of states adjacent to x.

[1]maria.jose.jimenez@upc.edu

The *combinatorial Laplacian* operator, or simply the *Laplacian*, of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) - u(y)\big),$$

for $x \in V$. It is well known that the Laplacian is self-adjoint and positive semidefinite. Additionally, $\mathcal{L}(u) = 0$ if and only if u is a constant function.

If we give an order on the vertex set V, then functions can be identified with vectors in $\mathbb{R}^n$, and operators can be identified with square n-matrices. Hence, suppose that $V = \{s_1, s_2, \dots, s_n\}$, then we will consider $c_{ij} = c(s_i, s_j)$. Every $u \in \mathcal{C}(V)$ is identified with $\big(u(s_1), u(s_2), \dots, u(s_n)\big)^{\mathsf{T}} \in \mathbb{R}^n$ and the combinatorial Laplacian with the symmetric irreducible matrix

$$\mathsf{L} = \begin{bmatrix} k_1 & -c_{12} & \dots & -c_{1n} \\ -c_{12} & k_2 & \dots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1n} & -c_{2n} & \dots & k_n \end{bmatrix},$$

where $k_i = k(s_i)$, $i = 1, \dots, n$. This matrix is diagonally dominant and, hence, it is positive semidefinite. Moreover, it is singular and 0 is a simple eigenvalue whose associated eigenvector is constant, $\mathsf{L}\mathsf{1} = \mathbf{0}$, where $\mathsf{1}$ is the all ones vector.

For the reader's convenience, vectors will be boldfaced. In particular, $\mathbf{k} = \big(k_1, k_2, \dots, k_n\big)^{\mathsf{T}}$ is the degree column-vector for Γ. Moreover, given a matrix A and a vector $\mathbf{v}$, we denote by A_d the diagonal matrix whose diagonal elements are a_{ii} and by $\mathsf{D}_{\mathbf{v}}$ the diagonal matrix whose diagonal elements are given by the elements of $\mathbf{v}$.

2.1 The MFPT

The short-term behavior of a random walk is modelled by the *mean first passage time* m_{ij}, for $i, j = 1, \dots, n$, $i \neq j$, which gives the expected number of time-steps $t \geq 1$, before the system reaches s_j, if it starts in s_i,

$$m_{ij} = E\big[t \mid X_t = s_j, X_0 = s_i\big],$$

where $E[\bullet]$ denotes the expected value of $\bullet$. It is well known [6] that, for $i \neq j$, $1 \leq i, j \leq n$,

$$m_{ij} = p_{ij} + \sum_{k \neq j} p_{ik}\big(m_{kj} + 1\big) = 1 + \sum_{k \neq j} p_{ik} m_{kj}. \tag{1}$$

Besides, the *mean recurrence time for state* s_i, denoted by m_{ii}, is the expected number of time steps before we return to s_i for the first time, for any $i = 1, \dots, n$. The mean recurrence time for state s_i also verifies Equation (1). If we define J as the matrix of order n with all entries equal to 1, we can write (1) in matrix form as in [2],

$$\mathsf{M} = \mathsf{J} + \mathsf{P}\mathsf{M} - \mathsf{P}\mathsf{M}_d. \tag{2}$$

Therefore, $m_{ii} = \dfrac{1}{\pi_i}$, since multiplying both sides of (2) by $\boldsymbol{\pi}^{\mathsf{T}}$, we obtain $\mathbf{0}^{\mathsf{T}} = \boldsymbol{\pi}^{\mathsf{T}}\left(\mathsf{J} - \mathsf{P}\mathsf{M}_d\right)$ or $\mathbf{0}^{\mathsf{T}} = \mathsf{1}^{\mathsf{T}} - \boldsymbol{\pi}^{\mathsf{T}}\mathsf{M}_d$.

We can use this last expression and Equation (2) to obtain the matrix expression for the MFPT

$$(\mathsf{I} - \mathsf{P})\mathsf{M} = \mathsf{J} - \mathsf{P}\mathsf{D}_{\boldsymbol{\pi}}^{-1}. \tag{3}$$

System (3) has solution because each column of $\mathsf{J} - \mathsf{P}\mathsf{D}_{\boldsymbol{\pi}}^{-1}$ belongs to $\boldsymbol{\pi}^{\perp}$ and the solution is unique up to a multiple of $\boldsymbol{\pi}$. Usually, generalized inverses of $\mathsf{I} - \mathsf{P}$ have been used in the literature to solve the above system, see for instance [3]. Remember that if A is any $m \times n$ singular matrix, a generalized inverse or 1-inverse of A is any matrix X such that $\mathsf{AXA} = \mathsf{A}$. For any 1-inverse $\widetilde{\mathsf{G}}$ of $\mathsf{I} - \mathsf{P}$, we get that

$$\mathsf{M} = \widetilde{\mathsf{G}}\big(\mathsf{J} - \mathsf{P}\mathsf{D}_{\boldsymbol{\pi}}^{-1}\big) + \mathsf{1}\boldsymbol{\alpha}^{\mathsf{T}},$$

being $\boldsymbol{\alpha}$ a constant vector.

We show next the expression of MFPT in terms of a 1-inverse of the combinatorial Laplacian L, instead of using 1-inverses of of $\mathsf{I} - \mathsf{P}$.

Proposition 1 *Let Γ be a connected network and G a 1-inverse of L, then the mean first passage time matrix M can be written as*

$$\mathsf{M} = \mathsf{G}\mathsf{D}_{\mathbf{k}}\mathsf{J} - \mathsf{J}\big(\mathsf{G}\mathsf{D}_{\mathbf{k}}\mathsf{J}\big)_d + \mathrm{vol}(\Gamma)\Big(\mathsf{D}_{\mathbf{k}}{}^{-1} - \mathsf{G} + \mathsf{J}\mathsf{G}_d\Big).$$

2.2 The Kemeny's constant

The well-known quantity $K \equiv \sum_{j \in V} m_{ij}\pi_j$ represents the time for reaching a random state s_j, starting from an initial state s_i according to the stationary distribution $\boldsymbol{\pi}$. It is a very curious fact that K does not depend on s_i, and hence the name *Kemeny's constant*. In a matrix-vector form, it is written as $\mathsf{M}\boldsymbol{\pi} = K\mathsf{1}$.

Our aim now is to express the Kemeny's constant using G, a 1-inverse of the combinatorial Laplacian.

Proposition 2 *If G is a 1-inverse of L such that $\mathsf{G}\mathbf{k} = g\mathsf{1}$, the Kemeny's constant is given by*

$$K = 1 - g + \mathrm{tr}(\mathsf{G}\mathsf{D}_{\mathbf{k}}).$$

3 Conclusions

In the field of random walks, the mean first passage time matrix and the Kemeny's constant allow us to deepen into the study of networks. For a transition matrix P, we can observe in the literature how the authors characterize mean first passage time using generalized inverses of $\mathsf{I} - \mathsf{P}$. In this work, we provide alternative expressions for fundamental parameters in the framework of random walks that involve generalized inverses of the combinatorial Laplacian, which is a symmetric singular M-matrix.

The results obtained in this manuscript parallel the ones obtained by Hunter in his big production (see [4] as an example) or the ones of other authors in [8, 9, 10].

Acknowledgments

This work has been partly supported by the Spanish Research Council (Ministerio de Ciencia e Innovación) under project PID2021-122501NB-I00 and by the Universitat Politècnica de Catalunya under funds AGRUPS 2023.

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