Embedded stable methods for the adaptive integration of stiff stochastic differential equations

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1 Introduction

Stochastic Differential Equations (SDEs) arise as natural mathematical models for describing random processes in a variety of application areas. Examples include blood clotting systems, cellular energetics, stochastic annealing, electrical activity of neural masses and noisy oscillators in a diversity of physical systems ([6], [8], [7]). In several practical situations, it is important that the trajectories (the sample paths) of the numerical approximations be close to the strong solution of an SDE. These direct simulations of the trajectories can provide considerable insight into the qualitative behavior and dynamics of the SDE ([1], [2], [3]). In general, the addition of noise to a deterministic system can drastically modify its deterministic dynamics. For instance, small noisy perturbation of a deterministic system can make bifurcation ill defined in a region near the critical value, can make trajectories flip backwards and forwards between different steady-states or can cause a crossing of attraction domains of stable equilibrium points. Therefore, the numerical integration of such stochastic systems should be done with care.

There is a variety of numerical integrators for SDEs, see, e.g., extensive surveys and comparative studies by simulations in ([6], [8], [9]). It is well-known that an appropriate trade-off among the basic requirements of stability, high order of convergence, and low computational effort of a numerical integrator is difficult to achieve in general. From the stability viewpoint, the common approach to construct integrators with large stability regions is through implicit methods. In the better case, some of them satisfy the elemental A-stability criterion, but at expense of a high computational cost. Hence, explicit methods are preferable for providing high order of convergence with lower algorithmic complexity. In practice, there are many examples of SDEs with bounded trajectories in which, for any fixed step-size of the time discretization, the numerical solution becomes explosive when the initial value is in a certain region of the phase space ([4], [5]). On the other hand adaptive time-stepping algorithms for the integration of SDEs are more efficient than their fixed step-size counterparts in terms of accuracy, stability and computational cost. Although the use of adaptive schemes for the approximation of ODEs are well-established, for SDEs are still on their initial stages of evolution. This is the case of variable step-size methods for additive noise SDEs.

The main purpose of this work is to introduce explicit embedded schemes for the variable stepsize integration of stiff systems of additive noise SDEs. The construction of the schemes is carried out by combining the local linearization approach with a suitably adapted Padê method resulting in stable embedded integrators. By using these integrators in the adaptive algorithm

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excessive restriction on the value of the stepsizes -due to stability considerations- is avoided, which implies a significant reduction of the number of time-steps used and thus, of the overall computation cost. Details on the efficient implementation of the adaptive algorithm are discussed and numerical experiments are presented to illustrate the practical performance of the proposed method.

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