Resolution of Nonlinear Systems with an Iterative Parametric Family

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1 Introduction

In this work we studied the following parametric family, with its iterative expression:

$$y_k = x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots$$
$$x_{k+1} = x_k - \frac{4}{4\beta(u_k-1)^2 + (7-3u_k)u_k}\frac{f(x_k)}{f'(y_k)},$$

Where x_0 is an initial estimate, β is a real or complex parameter and $u_k = \frac{f'(x_k)}{f'(y_k)}$.

The aim of our project is to examine every aspect of our family, in order to assess the efficiency of our method in solving nonlinear equations. If you like analogies, we are going to "squeeze" this family until we cannot get more relevant information.

We will analyze the convergence of this Iterative Parametric Family and adapt this method for systems of nonlinear equations.

Additionally, dynamic tools will be used and developed for both scalar and multivariable analysis. Finally, we will solve applied problems to validate the results of the dynamic study.

2 Analyzed Aspects

2.1 Order of Convergence of our Method

With the help of *Wolfram Mathematica 13.2* software we calculated the order of our method for the scalar version, since *Mathematica* does not respect the non-conmutativity of operators.

We discovered that, for a specific value of β, this method exhibits 4th-order convergence. In all other instances, it remains a 3rd-order family.

The final expression of our Taylor Development used to determine the order of convergence is:

$$e_{k+1} = \frac{1}{9}(16\beta-3)C_2^2e^3 + \frac{1}{27}e^4\left((51-176\beta)C_2^3 + 3(64\beta-21)C_3C_2 + 3C_4\right) + O\left(e^5\right)$$

Being $e_k = x_k - \bar{x}$ and $C_i = \frac{f^i(\bar{x})}{f'(\bar{x})i!}$.

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2.2 Non Linear System Version and its Order

Fortunately, our method can be directly extended to equation systems. The reason is quite simple: all the terms in the denominator of our iterative expression are derivatives. Therefore, in the vectorial case, these derivatives can be represented as gradients, which can then be transformed into matrices, specifically inverse matrices, as they are in the denominator. The extension of our family is:

$$\begin{aligned} y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x_k)]^{-1}F(x_k), \quad k = 0, 1, \ldots \\ x^{(k+1)} &= x^{(k)} - 4[4\beta(U_k - I_n)^2 + (7I_n - 3U_k)U_k]^{-1}([F'(y_k)]^{-1}F(x_k)), \end{aligned}$$

Where $x^{(0)}$ is an initial estimate vector, β is a parameter (real or complex) and $U_k = [F'(y_k)]^{-1}F'(x_k)$. We consider the order of our operations from right to left.

When discussing the convergence order of a nonlinear method, we cannot rely on *Mathematica* assistance because, as mentioned earlier, this approach does not preserve the non-commutativity of operators. Therefore, the convergence order of our method could be altered by such transformations. However, we found that this is not the case in this instance. The final expression is:

$$e_{k+1} = x^{k+1} - \bar{x} = x^k - \bar{x} - 4[(7I - 3U_k)U_k + 4\beta(U_k - I)^2]^{-1}([F'(y^{(k)})]^{-1}[F(x^{(k)})]) = \frac{1}{9}(-3 + 16\beta)\, e_k^3 + O(e_k^4)$$

Where I_n is the identity matrix of size n. We observed that, when using the same value that yields a 4th-order method in the scalar case, $\beta = \frac{3}{16}$, we achieved a 4th-order method. In all other cases, it remained a 3rd-order nonlinear method.

2.3 Computational Efficiency and Ostrowski Index

Computational Efficiency Index (IC):

$$IC = p^{\frac{1}{d+op}}.$$

where:

p : Convergence Order.

d : Number of functional evaluations per iteration.

op : Number of products/divisions per iteration.

Ostrowski Efficiency Index (IO):

$$IC = p^{\frac{1}{d}}.$$

where:

p : Convergence Order.

d : Number of functional evaluations per iteration.

As demonstrated previously, different values of β result in different Efficiency Index values. Additionally, we compared the Efficiency Indexes of our method with those of the Newton Method. We did this because the Newton Iterative Method is renowned for its exceptional performance in solving nonlinear systems, and we aim to determine if our method has any relevance in this Area of Mathematics .

The expressions for the Newton Method are:

$$\text{ICN} = 2^{\frac{1}{\frac{1}{3}n^3+2n^2+\frac{2}{3}n}}$$

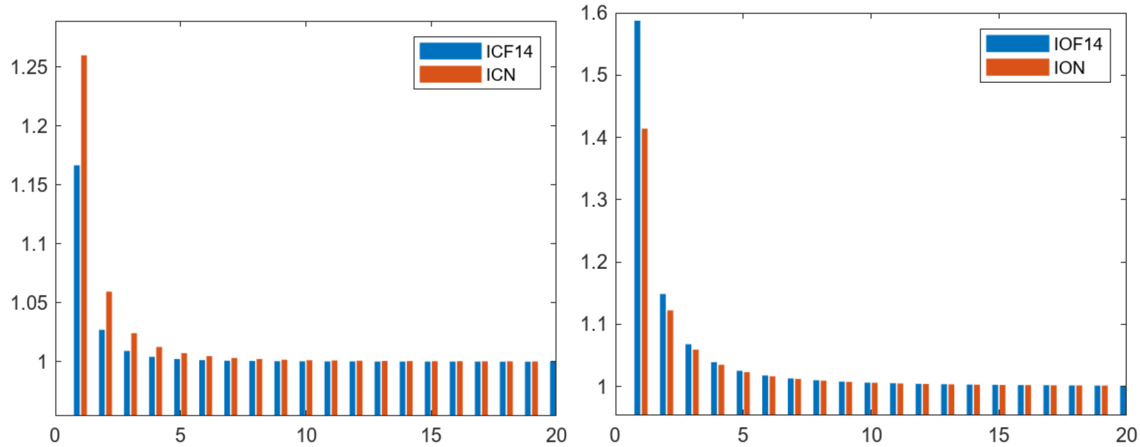
$$\text{ION} = 2^{\frac{1}{n^2+n}}$$

The expression for our method for $\beta = \frac{3}{16}$

$$ICF14 = 4^{\frac{1}{4n^3+5n^2}} = 2^{\frac{1}{2n^3+2.5n^2}}$$

$$IOF14 = 4^{\frac{1}{2n^2+n}} = 2^{\frac{1}{n^2+0.5n^2}}$$

Considering n as the number of unknowns, we performed a comprehensive graphical comparison. Here, **ICF14** or **IOF14** denote the indexes of our method, while **ICN** or **ION** represent the indexes of the Newton Method.



(a) Comparison IC $\beta = \frac{3}{16}$.

(b) Comparison IO $\beta = \frac{3}{16}$.

2.4 Stability of Our Method (Dynamic Analysis)

For our Dynamical study, we analyzed the 2^{nd} grade polynomial:

$$p(x) = (x-a) * (x-b)$$

After applying our method we obtain:

$$F14 = x + \frac{(-a+x)(-b+x)(3a^2+2ab+3b^2-8(a+b)x+8x^2)^2}{(16\beta(a-x)^2(b-x)^2+3(a+b-2x)^2(3a^2-ab+3b^2-5(a+b)x+5x^2)(a+b-\frac{2((b-4x)x+a(2b+x))}{3(a+b-2x)})}$$

Where **F14** is the fixed-point operator associated to the iterative expression. We applied the Möbius transformation ($h(z) = \frac{z-a}{z-b}$)to obtain a simplified result:

$$R = \frac{x^3(-3+16\beta+6x+9x^2)}{9+6x-3x^2+16\beta * x^2}$$

Here, we encountered a crucial aspect of our study. Up until this point, we believed that the optimal value for the parameter was $\beta = \frac{3}{16}$. However, this value fails the Cayley Test. For $\beta = \frac{3}{16}$:

$$R = \frac{x^4(2+3x)}{3+2x}$$

On the other hand, for $\beta = \frac{3}{4}$, we obtained an operator that did pass the Cayley Test. This implies our fixed points correspond to the roots of the polinomial:

$$R = x^3$$

Fixed Points

Studying our family for a general parameter β we got the following fixed points $R(x) = x$.

$$\begin{aligned} x &= -1 \\ x &= 0 \\ x &= 1 \\ x &= \frac{1}{3}\left(-1 - 2i\sqrt{2}\right) \\ x &= \frac{1}{3}\left(-1 + 2i\sqrt{2}\right) \\ x &= \infty \end{aligned}$$

In other words, we have 4 strange fixed points (which are not the roots, 0 or ∞). We characterise the stability of the fixed points with the derivative of our operator $R'(x)$:
$|R'(z^*)| < 1 \Rightarrow$ Attractor
$|R'(z^*)| > 1 \Rightarrow$ Repulsive
$|R'(z^*)| = 1 \Rightarrow$ Neutral or Parabolic
$|R'(z^*)| = 0 \Rightarrow$ Superattractor

After studying for which values of β a peculiar point is attractive or repulsive, we constructed the following plane. The XY-plane represents the complex values of β, while the Z-axis denotes the value of the $R'(x)$ operator at each peculiar fixed point. Consequently, in the orange zone, it is less likely that our method converges to the roots:

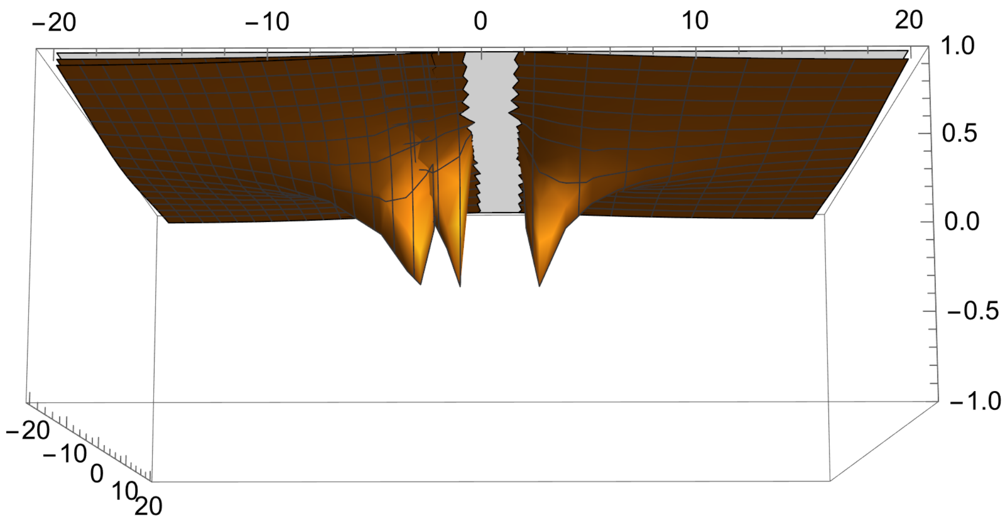


Figure 2: Total Stability in the Gray Zone.

Critical Points

Having studied the fixed points, we studied the critical points, where $R'(x) = 0$, and thus our family becomes superattractive:

- x=0 (as seen before)
- x=∞ (as seen before)

- $cr_1 = \frac{1}{9}\Big(-\frac{2\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}}{\sqrt{(16\beta-3)^2}} + \frac{16\beta}{3-16\beta} + \frac{15}{3-16\beta}$ $-2\sqrt{\frac{-12288\beta^4-30208\beta^3-432\beta^2+8\left(2\sqrt{(16\beta-3)^2}\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}+783\right)\beta+15\sqrt{(16\beta-3)^2}\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}-945}{(16\beta-3)^3}}\Big)$
- $cr_2 = \frac{1}{9}\Big(-\frac{2\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}}{\sqrt{(16\beta-3)^2}} + \frac{16\beta}{3-16\beta} + \frac{15}{3-16\beta}$ $+2\sqrt{\frac{-12288\beta^4-30208\beta^3-432\beta^2+8\left(2\sqrt{(16\beta-3)^2}\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}+783\right)\beta+15\sqrt{(16\beta-3)^2}\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}-945}{(16\beta-3)^3}}\Big)$
- $cr_3 = \frac{1}{9}\Big(+\frac{2\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}}{\sqrt{(16\beta-3)^2}} + \frac{16\beta}{3-16\beta} + \frac{15}{3-16\beta}$ $-2\sqrt{\frac{-12288\beta^4-30208\beta^3-432\beta^2+8\left(2\sqrt{(16\beta-3)^2}\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}+783\right)\beta+15\sqrt{(16\beta-3)^2}\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}-945}{(16\beta-3)^3}}\Big)$
- $cr_4 = \frac{1}{9}\Big(+\frac{2\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}}{\sqrt{(16\beta-3)^2}} + \frac{16\beta}{3-16\beta} + \frac{15}{3-16\beta}$ $+2\sqrt{\frac{-12288\beta^4-30208\beta^3-432\beta^2+8\left(2\sqrt{(16\beta-3)^2}\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}+783\right)\beta+15\sqrt{(16\beta-3)^2}\sqrt{-768\beta^3+3088\beta^2-2472\beta+441}-945}{(16\beta-3)^3}}\Big)$

We only have 2 critical free independent points because:

- $cr_3 = 1/cr_4$
- $cr_1 = 1/cr_2$

An interesting observation in this section is that, for $\beta = \frac{3}{4}$, we do not obtain any free critical points. That is another aspect in favor of using this value of the parameter.

We created an illuminating graphic to clarify the concepts regarding these lengthy expressions and therefore avoid messy outcomes.

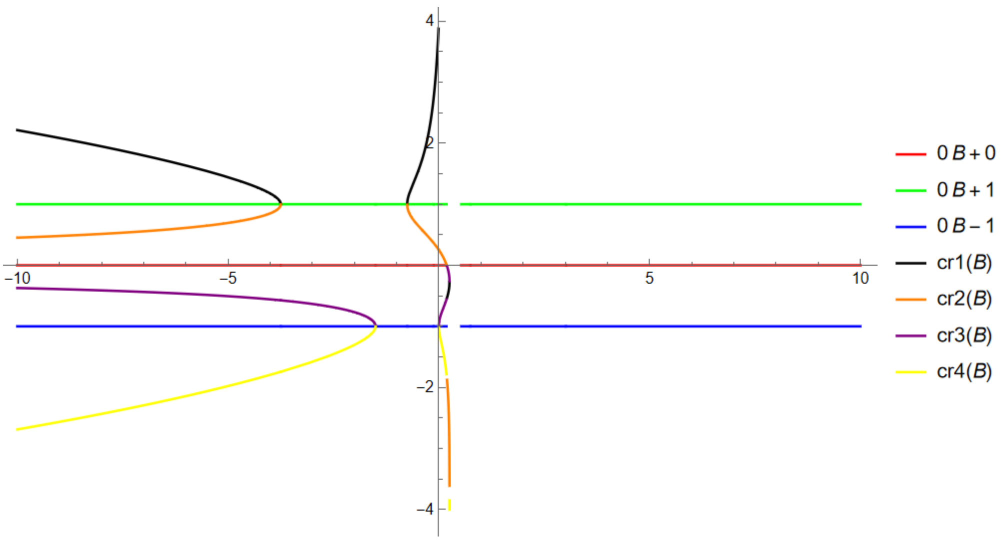


Figure 3: X-Axis are real values of β.
Y-Axis are real values of cr_i

2.5 Study of the Dynamic Plane

We studied the different Dynamic Planes that we could generate by giving our parameter different values. These planes function in the following way. For a specific value of the parameter, we generated a colorful plane that indicates whether our method is likely to converge to the roots or not.

- Orange and Blue: It converges to a root.
- Black: It does not converge.
- Other colour: It converges to a strange fixed point.

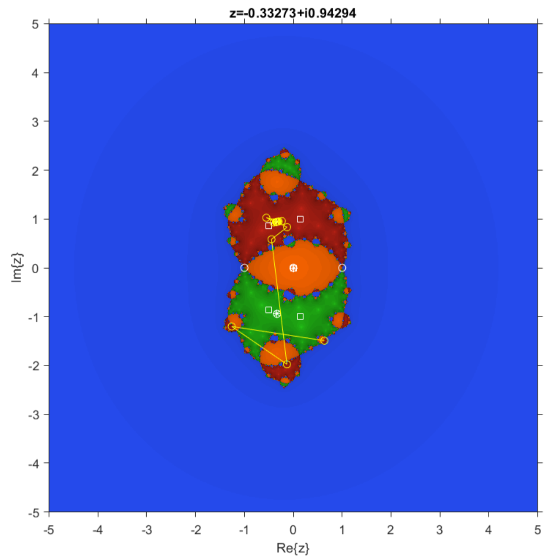
We can illustrate this with an example to provide a concrete understanding of how it works. In addition, based on theory, we know that all critical points lie within a region of convergence.

$\beta = 2$ **case**

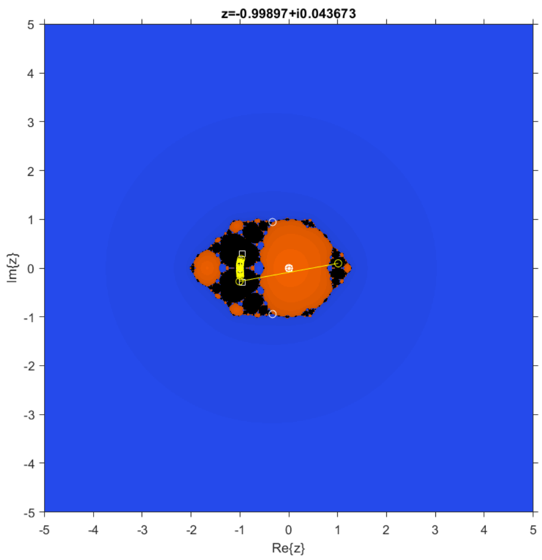
A black region does not exist and, therefore, our method will always converge. Nonetheless, it could converge to a strange fixed point. The critical points are:

- $cr_1 = 0.1363 + 0.9907i$
- $cr_2 = \overline{cr_1}$
- $cr_3 = -0.4965 + 0.8680i$
- $cr_4 = \overline{cr_3}$

cr_1 and cr_3 are in the red region, which corresponds to the points which converge to the strange fixed point $\frac{1}{3}\left(-1 + 2i\sqrt{2}\right)$. Its conjugates, cr_2 and cr_4, belong to the red region of the strange fixed point $\frac{1}{3}\left(-1 - 2i\sqrt{2}\right)$.



$\beta = -3/4$ **case**



The black region belongs to the orbit with period -0.999 ± 0.0437i. In this case, cr_1 and cr_2 are not symbolized with a square in the image, but they are in the black region. The critical points are:

- $cr_1 = 1$
- $cr_2 = 1$
- $cr_3 = 0.9556 + 02948i$
- $cr_4 = \overline{cr_3}$

The points cr_1 and cr_2 are located in the basin of attraction of the periodic orbit. In this case, all critical points are found in the black region, indicating poor stability. On the other hand, strange fixed points are not attractors, which is a point in favor.

2.6 Study of the Parametric Planes

These planes are useful for assessing the stability of the method in terms of the value of the parameter beta, where the XY-Plane is the complex plane that represents the values that β can take. The more red is the region, the more likely we converge to a root, the more black is the region, the more likely is not to convergence to a root.

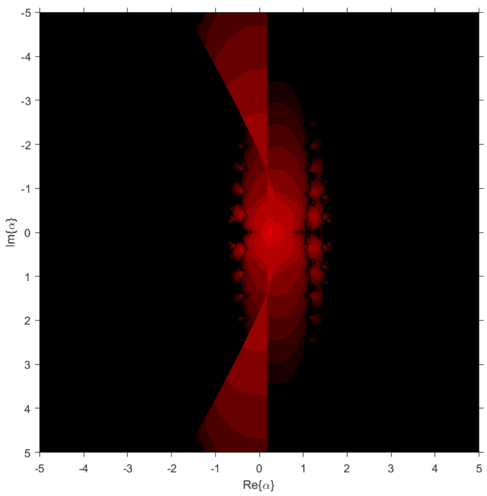


Figure 4: Parametric plane for our second independent critical point $cr_2 = \frac{1}{cr_1}$

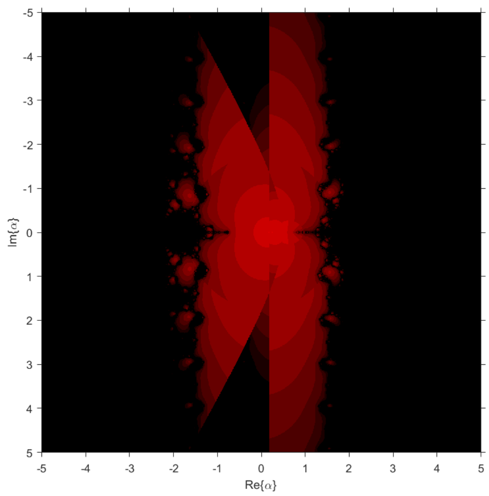


Figure 5: Parametric plane for our fourth independent critical point $cr4 = \frac{1}{cr_3}$

3 Results

3.1 Data of Execution of our Family

In this work, it is essential to test our method before proceeding with any further analysis or study. Therefore, we executed our method considering the results obtained previously.
We tested our method with this particular example to verify its general validity.

$$F(x_1, x_2) = (e^{x_1}e^{x_2} + x_1\cos(x_2), x_1 + x_2 - 1)\,,$$

One solution to this system is :

$$\bar{x} \approx \begin{pmatrix} 3.47063096 \\ -2.47063096 \end{pmatrix}.$$

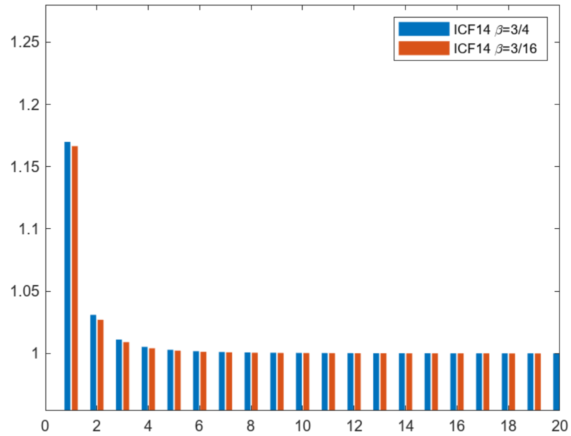
Table 1: Numerical tests of our family on the example system

β	$x^{(0)}$	**Iter**	**ACOC**	$\|F(x^{k+1})\|$	**Sol**
3/4	$[3; -1]$	5	3.2112	0.0	[3.4706;-2.4706]
3/16	$[3; -1]$	5	3.7374	0.0	[3.4706;-2.4706]
1+2i	$[3; -1]$	7	2.9730	1.7286e-38	[3.4706;-2.4706]
-3/2	$[3; -1]$	5	2.5376	4.4409e-16	[3.4706;-2.4706]
$-5-5i$	$[3; -1]$	2905	0.9993	NaN	[NaN;NaN]
$3+2i$	$[3; -1]$	57	NaN	NaN	[NaN;NaN]
3/4	$[10; -10]$	5	3.0801	1.0214e-14	[11.7624;-10.7624]
3/16	$[10; -10]$	5	2.0256	1.0214e-14	[11.7624;-10.7624]
1+2i	$[10; -10]$	7	1.6533	1.0214e-14	[11.7624;-10.7624]
-3/2	$[10; -10]$	6	Inf	7.6383e-14	[59.1654;-58.1654]
$-5-5i$	$[10; -10]$	3000	0.997	7.7636	[NaN;NaN]
$3+2i$	$[10; -10]$	7	3.0110	1.0214e-14	[11.7624;-10.7624]

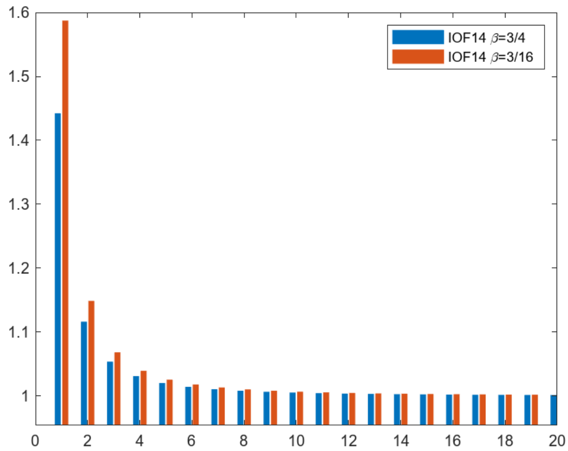
3

4 Conclusions

In conclusion, after conducting the entire study and all the numerical tests, we determined that the ideal value for our method was $\beta = 3/16$. As we have observed, it exhibits the best behavior when solving non-linear systems, has an order of 4, lies within the red zone (indicating good stability), and its computational efficiency is only slightly lower than that of $\beta = 3/4$, and so would hardly make a difference. $\beta = 3/4$ would be the second-best option for our parameter. For $\beta = 3/16$, our method would be almost as efficient as the Newton method.



(a) Comparison of I.C for $\beta = 3/4$ and $\beta = 3/16$



(b) Comparison I.O for $\beta = 3/4$ and $\beta = 3/16$

A noteworthy aspect of our work is that values of β promoting convergence in second-degree polynomial roots also support convergence in other systems. This enables cautious extrapolation of findings from simple polynomials, such as $p(x) = (x - a)(x - b)$, to potentially more complex nonlinear systems. However, exceptions may arise.

Acknowledgments

We wish to extend our sincere gratitude to our esteemed teachers, Juan Ramón Torregrosa and Alicia Cordero. Their invaluable guidance and assistance were key to resolve numerous doubts and questions that arose throughout the duration of this project. Furthermore, they imparted comprehensive explanations of the underlying theory. The privilege of presenting this paper at the congress is a direct result of the opportunity they afforded us to participate in this esteemed event.

References

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[3] ACOC: Approximated Computational Order of Convergence