On the spectrum of Jacobi almost-Toeplitz matrices. Application to Jacobi bisymmetric matrices

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1 Introduction

The inverse eigenvalue problem for symmetric matrices consists on determining whether for an ordered list of n real numbers there exists a symmetric and irreducible matrix of order n whose spectrum coincides with the given list. Within this general problem, the case of bisymmetric Jacobi matrices occupies a central place, since for each strictly monotone list, there is a unique bisymmetric Jacobi matrix that realizes it. The term *Jacobi matrix* refers to a symmetric tridiagonal matrix whose secondary diagonal has negative entries, so being irreducible Z-matrices. The term *bisymmetric* refers to matrices that are symmetric with respect to their two main diagonals. Regardless of their interest in fields such as mechanics or statistics, families of this type of matrices whose spectrum is known are often used as tests for different recovery algorithms of the coefficients of the matrices from spectral data. Unfortunately, there are very few families of bisymmetric Jacobi matrices whose spectrum is known. Recently, bisymmetric Jacobi matrices whose eigenvalues form a linear or quadratic monotone progression have been characterized, see [5], thus unifying many different works over the last hundred years. For the case of periodic coefficients, the main results in the literature refer mainly to low periods, up to 3, see for instance [1, 6, 7], but in these works the bisymmetric case is not considered.

Our spectral analysis for Jacobi matrices is based on difference equations and the boundary (Sturm-Liouville) problems associated with them, see the general results in [3]. Then we can incorporate to this study our results about second order difference equation with periodic coefficients, see [2], obtaining so a general perspective of this kind of matrices. Finally, we consider the low periodic case and then describe explicitly the rich structure of bisymmetric Jacobi matrices.

2 Spectral analysis of Jacobi matrices and difference equations

In the sequel for any $n \in \mathbb{N}^*$, vectors in $\mathbb{R}^{n+1}$ are denoted as $\mathsf{v} = (v_0, \dots, v_n)$, (always with the subindex starting at $k = 0$). With this notation $\mathsf{v} > 0$ means that $v_k > 0$, $k = 0, \dots, n$ and the set of such vectors is denoted as $(\mathbb{R}^+)^{n+1}$. In addition the null and the all ones vectors in $\mathbb{R}^{n+1}$ are denote by $\mathsf{0}_{n+1}$ and e_{n+1}, respectively. We drop the subindexes when it does not lead to confusion.

We call $\mathsf{z} \in \mathbb{R}^{n+1}$ *periodic with period* $p \in \mathbb{N}^*$, $1 \le p < n$ if it satisfies that $z_{p+k} = z_k$, for any $k = 0, \dots, n+1-p$. The subspace of periodic vectors in $\mathbb{R}^{n+1}$ with period p is denoted as $\mathbb{R}^{n+1}_p$.

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It is clear that $\dim(\mathbb{R}_p^{n+1}) = p$, since each periodic vector with period p is determined by its first p entries. Specifically, if $n+1 = mp + r$ with $m \in \mathbb{N}^*$ and $0 \le r < p$, then $z_{kp+j} = z_j$ for any $k = 0, \dots, m-1$ and $j = 0, \dots, p-1$ and when $r \ge 1$, $z_{mp+j} = z_j$ for any $j = 0, \dots, r-1$. We also consider the set $(\mathbb{R}^+)_p^{n+1} = (\mathbb{R}^+)^{n+1} \cap \mathbb{R}_p^{n+1} = \{\mathsf{z} \in \mathbb{R}_p^{n+1} : \mathsf{z} > 0\}$.

We call $\mathsf{z} \in \mathbb{R}^{n+2}$ *almost-periodic with period* $p \in \mathbb{N}^*$, $1 \le p < n$ if it satisfies that $z_{p+k} = z_k$, for any $k = 1, \dots, n-p$. The subspace of almost-periodic vectors in $\mathbb{R}^{n+2}$ with period p is denoted as $\mathbb{A}_p^{n+2}$ and then $\dim(\mathbb{A}_p^{n+2}) = p + 2$. Specifically, if $n+1 = mp + r$ with $m \in \mathbb{N}^*$ and $0 \le r < p$, then $z_{kp+j} = z_j$ for any $k = 0, \dots, m-1$ and $j = 1, \dots, p-1$ and when $r \ge 1$, $z_{mp+j} = z_j$ for any $j = 0, \dots, r-1$. In addition, we define the *periodicity values* $\sigma = z_0 - z_p$ and $\tau = z_{n+1} - z_r$. Of course, in any case, if $\mathsf{z} \in \mathbb{A}_p^{n+2}$, then $\mathsf{z} \in \mathbb{R}_p^{n+2}$ iff with the above notation $\sigma = \tau = 0$.

Given $m \in \mathbb{N}$, the vector $\mathsf{z} = (z_0, \dots, z_m) \in \mathbb{R}^{m+1}$ is called *centrosymmetric* when $z_{m-k} = z_k$, for all $k = 0, \dots, m$. It is clear that the vector z is centrosymmetric iff $z_{m-k} = z_k$ for any $k = 0, \dots, \lfloor \frac{m}{2} \rfloor$. Therefore, if $\mathbb{S}^{m+1}$ is the subspace of centrosymmetric vectors in $\mathbb{R}^{m+1}$, then $\dim(\mathbb{S}^{m+1}) = \lceil \frac{m+1}{2} \rceil$.

The main objective of this work is to describe the eigenvalues and eigenvectors of some clases of Jacobi matrices of order $n+2$. According with the previously established framework, here Jacobi matrix means a tridiagonal matrix of the form

$$\mathsf{J}(\mathsf{a},\mathsf{b}) = \begin{bmatrix} a_0 & -b_0 & 0 & \cdots & 0 \\ -b_0 & a_1 & -b_1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_n & -b_n \\ 0 & \cdots & 0 & -b_n & a_{n+1} \end{bmatrix} \tag{1}$$

where $\mathsf{a} = (a_0, a_1, \dots, a_n, a_{n+1}) \in \mathbb{R}^{n+2}$ and $\mathsf{b} = (b_0 \dots, b_n) \in (\mathbb{R}^+)^{n+1}$.

The Jacobi matrix $\mathsf{J}(\mathsf{a},\mathsf{b})$ is called *bisymmetric* when a and b are both centrosymmetric; that is, $a_{n+1-k} = a_k$, $k = 0, \dots, n+1$ and $b_{n-k} = b_k$, $k = 0, \dots, n$. Therefore, a bisymmetric Jacobi matrix is determined by $\lceil \frac{n+1}{2} \rceil + \lceil \frac{n+2}{2} \rceil = n + 2$ values.

In the mathematical literature, the Jacobi matrix $\mathsf{J}(\mathsf{a},\mathsf{b})$ is called *p-Toeplitz* when its entries along the diagonals are vectors of period p; that is, when the pair $(\mathsf{a},\mathsf{b}) \in \mathbb{R}_p^{n+2} \times (\mathbb{R}^+)_p^{n+1}$, see for instance [1, 6]. Attending this terminology, if $1 \le p < n$, we call the Jacobi matrix $\mathsf{J}(\mathsf{a},\mathsf{b})$ *almost p-Toeplitz* when a is almost-periodic with period p and b is periodic with period p; that is, when $(\mathsf{a},\mathsf{b}) \in \mathbb{A}_p^{n+2} \times (\mathbb{R}^+)_p^{n+1}$. We call *Jacobi almost-Toeplitz* matrix a Jacobi matrix that is almost p-Toeplitz for some $p \ge 1$.

Given $\mathsf{f} \in \mathbb{R}^{n+2}$, when $\mathsf{J}(\mathsf{a},\mathsf{b})\mathsf{v} = \mathsf{f} \in \mathbb{R}^{n+2}$, then $\mathsf{v} \in \mathbb{R}^{n+2}$ is a solution of the system

$$\begin{aligned} a_0 v_0 - b_0 v_1 &= f_0, \\ -b_{k-1} v_{k-1} + a_k v_k - b_k v_{k+1} &= f_k, \quad k = 1, \dots, n, \\ -b_n v_n + a_{n+1} v_{n+1} &= f_{n+1}. \end{aligned} \tag{2}$$

In particular, when $\mathsf{f} = \mathsf{0}$ we refer to the above linear system (2) as *homogeneous*.

We are interested in describing those real numbers, $\lambda \in \mathbb{R}$, and those vectors, $\mathsf{v} \in \mathbb{R}^{n+2}$, such that $\mathsf{J}(\mathsf{a},\mathsf{b})\mathsf{v} = \lambda \mathsf{v}$ or equivalently, such that they are the solutions of the homogeneous system $\mathsf{J}(\mathsf{a} - \lambda \mathsf{e}, \mathsf{b})\mathsf{v} = \mathsf{0}$.

If for any $\lambda \in \mathbb{R}$ we consider the subspace

$$\mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda) = \{\mathsf{v} \in \mathbb{R}^{n+2} : \mathsf{J}(\mathsf{a},\mathsf{b})\mathsf{v} = \lambda \mathsf{v}\} = \ker\big(\mathsf{J}(\mathsf{a} - \lambda \mathsf{e}, \mathsf{b})\big),$$

then λ is called *an eigenvalue of* $\mathsf{J}(\mathsf{a},\mathsf{b})$ when $\dim \mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda) \ge 1$ and then $\mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda)$ is called *eigenvector subspace* corresponding to λ. Therefore, the eigenvalues of $\mathsf{J}(\mathsf{a},\mathsf{b})$ are the roots of $P_{\mathsf{a},\mathsf{b}}(x) = \det\big(\mathsf{J}(\mathsf{a} - x\mathsf{e}, \mathsf{b})\big)$, the *characteristic polynomial* of $\mathsf{J}(\mathsf{a},\mathsf{b})$.

As we can observe, in system (2), the equations for the boundary nodes are different from the equations corresponding to interior nodes of the path and this is one of the main characteristics of our approach. In fact, we can view the interior equations as a difference equation on the path, and the equations at the boundary as two boundary conditions. Since each boundary condition only involves the boundary nodes and its adjacent nodes, they represent the discrete version of *separated*, also known as *Sturm-Liouville*, boundary conditions. This is the way we understand and study Jacobi matrices, interpreting them from (second order) discrete linear boundary value problems. Given $\mathsf{f} \in \mathbb{R}^{n+2}$, the equation $\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = \mathsf{f}$ on $\{1,\dots,n\}$, where the linear operator $\Delta_{\mathsf{a},\mathsf{b}} \colon \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^n$, named *discrete Schrödinger operator with coefficients* a *and* b, is defined by the three-term equation

$$\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{v})_k = -b_{k-1}v_{k-1} + a_k v_k - b_k v_{k+1}, \;\; k = 1,\dots,n, \tag{3}$$

is known as the *Schrödinger (difference) equation* with coefficients a and b and data f. Since in this work we only consider the homogeneous Schrödinger difference equations; that is, when $\mathsf{f} = \mathsf{0}$, in the sequel we use Schrödinger equation as synonymous of homogeneous Schrödinger equation.

The operator $\Delta_{\mathsf{a},\mathsf{b}}$ does not depend on the boundary entries, a_0 and a_{n+1}, of vector a. Now we clarify their role by considering the *boundary operators*, $\mathcal{B}^0_{\mathsf{a},\mathsf{b}}, \mathcal{B}^{n+1}_{\mathsf{a},\mathsf{b}} \colon \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$ defined as

$$\mathcal{B}^0_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = a_0 v_0 - b_0 v_1 \quad \text{and} \quad \mathcal{B}^{n+1}_{\mathsf{a},\mathsf{b}}(x) = a_{n+1}v_{n+1} - b_n v_n \tag{4}$$

that depend only on the coefficients a_0, a_{n+1}, b_0 and b_n.

The homogeneous *Sturm-Liouville Problem* associated with a and b consist on finding those $\mathsf{v} \in \mathbb{R}^{n+2}$ satisfying

$$\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = \mathsf{0} \quad \text{and} \quad \mathcal{B}^0_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = \mathcal{B}^{n+1}_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = 0. \tag{5}$$

If $\mathcal{V}_{\mathsf{a},\mathsf{b}} \subset \mathbb{R}^{n+2}$ is the set of solutions of (5), then it is clear that $\mathcal{V}_{\mathsf{a},\mathsf{b}}$ is a vector subspace of $\ker(\Delta_{\mathsf{a},\mathsf{b}})$, the nullity of $\Delta_{\mathsf{a},\mathsf{b}}$. The Sturm-Liouville problem is called *regular* when $\dim \mathcal{V}_{\mathsf{a},\mathsf{b}} = 0$ and *singular* otherwise.

The following Lemma resume the well-known properties of the solutions for the homogeneous Schrödinger equation; that is, the elements in $\ker(\Delta_{\mathsf{a},\mathsf{b}})$. All results and motivations can be found in [2, 3].

Lemma 2.1 *When the Sturm-Liouville problem* (5) *is singular, then* $\dim(\mathcal{V}_{\mathsf{a},\mathsf{b}}) = 1$ *and moreover,* $\mathcal{V}_{\mathsf{a},\mathsf{b}} = \operatorname{span}\{\mathsf{v}\}$*, where* v *is the unique solution of the homogeneous Schrodinger equation satisfying* $v_0 = b_0$, $v_1 = a_0$.

In view of the above result, given $\mathsf{a} \in \mathbb{R}^{n+2}$ and $\mathsf{b} \in (\mathbb{R}^+)n+1$, we call *fundamental solution* of the Schrödinger equation $\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{z}) = \mathsf{0}$ the unique $\mathsf{v} \in \mathbb{R}^{n+2}$ such that $\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = \mathsf{0}$ and moreover $v_0 = b_0$ and $v_1 = a_0$, which implies that $\mathcal{B}^0_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = 0$. So, the boundary value problem (5) is singular iff the fundamental solution of the homogeneous Schrödinger equation satisfies that $\mathcal{B}^{n+1}_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = 0$; that is, $a_{n+1}v_{n+1} = b_n v_n$ and then, $\mathcal{V}_{\mathsf{a},\mathsf{b}} = \operatorname{span}\{\mathsf{v}\}$.

We are ready to establish the main result about the eigenvalues and eigenvectors for the Jacobi matrix $\mathsf{J}(\mathsf{a},\mathsf{b})$.

Theorem 2.2 *Given* $\mathsf{a} \in \mathbb{R}^{n+2}$ *and* $\mathsf{b} \in (\mathbb{R}^+)^{n+1}$*, for any* $x \in \mathbb{R}$*, consider* $\mathsf{v}(x)$ *the fundamental solution for the homogeneus Schrödinder equation* $\Delta_{\mathsf{a}-x\mathsf{e},\mathsf{b}}(\mathsf{z}) = \mathit{0}$ *on* $\{1,\dots,n\}$*. Then, for any* $k = 0,\dots,n+1$*,* $v_k(x)$ *is a polynomial in* x *with degree* k *and the characteristic polinomial of* $\mathsf{J}(\mathsf{a},\mathsf{b})$ *is*

$$P_{\mathsf{a},\mathsf{b}}(x) = b_1 \cdots b_n\big((a_{n+1} - x)v_{n+1}(x) - b_n v_n(x)\big).$$

Therefore, $\lambda \in \mathbb{R}$ *is an eigenvalue of* $\mathsf{J}(\mathsf{a},\mathsf{b})$ *iff* $(a_{n+1} - \lambda)v_{n+1}(\lambda) = b_n v_n(\lambda)$ *and when this happens* $\mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda) = \operatorname{span}\{\mathsf{v}(\lambda)\}$.

After Theorem 2.2, it is clear that obtaining a closed formula for the eigenvalues of a Jacobi matrix is equivalent to give an expression for the fundamental solution of second order difference equations. So, the following question appears in a natural way:

For which coefficients $\mathsf{a} = (a_0, \dots, a_{n+1}) \in \mathbb{R}^{n+2}$ *and* $\mathsf{b} = (b_0, \dots, b_n) \in (\mathbb{R}^+)^{n+1}$ *we can obtain explicitly* $\mathsf{v}(x)$*, the fundamental solution of the homogeneous Schrödinger equation*

$$\Delta_{\mathsf{a}-x\mathsf{e},\mathsf{b}}(\mathsf{z})_k = -b_{k-1}z_{k-1} + (a_k - x)z_k - b_k z_{k+1} = 0,\ k = 1, \dots, n \tag{6}$$

for any $x \in \mathbb{R}$*?*

Recall that $\mathsf{v}(x)$ is characterized as the unique solution of equation (6) satisfying that $v_0(x) = b_0$ and $v_1(x) = a_0 - x$. Furthermore, to do this, the entry a_{n+1} of vector a has no role. It will only be relevant when we need to know if x is an eigenvalue of $\mathsf{J}(\mathsf{a}, \mathsf{b})$.

To treat to answer the above raised question we first identify some class of Schrödinger equations for which an explicit expression for its fundamental solution is avaible.

We start our findings with the easier, and very well-known case, when the coefficients of the homogeneous Schödinger equation (6) are constant. This means that $a_k = a$, $k = 1, \dots, n$ and $b_k = b > 0$, $k = 0, \dots, n$.

The analysis of the coefficients constant case will serve us a motivation and guide for the following developments. Although a very common method is to consider the *Binet basis*; that is, the two roots of the polynomial $-bx^2 + (a - \lambda)x - b$, we follow here another route that was also used, implicitly or explicitly, in previous works, see for instance [1, 2] to mention only a few.

If for given $x \in \mathbb{R}$ we define $q = \dfrac{a - x}{2b}$, the homogeneous Schrödinger equation (6) is equivalent, in the sense that it has the same solutions, to the *Chebyshev equation with coefficient* q

$$-z_{k-1} + 2qz_k - z_{k+1} = 0,\ k = 1, \dots, n. \tag{7}$$

Observe that if $\{P_k(x)\}_{k \in \mathbb{Z}}$ is a Chebyshev sequence of polynomials, see [?], then the vector $\mathsf{v} = \big(P_0(q), \dots, P_{n+1}(q)\big) \in \mathbb{R}^{n+2}$ is a solution of the Chebysev equation (7).

When the coefficients are non constant but they are periodic, then we can obtain its solutions in a similar way than in the case of constant coefficients. The authors are firmly convinced that this is the (technical) reason why so many papers on Jacobi matrices with periodic coefficients have appeared over time and also the reason for that Chebyshev polynomials play a fundamental role in the spectral analysis of this type of matrices.

Next we reproduce here the main results about linear second order difference equations with periodic coefficients with the aim of obtaining the fundamental solution of the corresponding homogeneous Schrödinger equation (6). All results related with difference equations with periodic coefficients can be found in [2].

We say that the Schrödinger operator $\Delta_{\mathsf{a},\mathsf{b}}$ *has periodic coefficients with period* p if $\mathsf{a} \in \mathbb{A}_p^{n+2}$ and $\mathsf{b} \in (\mathbb{R}^+)_p^{n+1}$. It is clear that in this case, for any $x \in \mathbb{R}$ we have that $\Delta_{\mathsf{a}-x\mathsf{e},\mathsf{b}}$ has also periodic coefficients with period p.

Given $n \in \mathbb{N}^*$ and $1 \le p < n$ we consider $m = \lfloor \frac{n+1}{p} \rfloor$, so $m \in \mathbb{N}^*$ and $n + 1 = mp + r$, where and $0 \le r < p$; that is, $n + 1 = r (\operatorname{mod} p)$. Notice that when $p = 1$, then $r = 0$, $m = n + 1$ and the case corresponds to a Schrödinger equation with constant coefficients, previously analized.

Although this scenario seems to be far away from the easiest one, we will show that it is not the case, since the main result in [2] establishes that irreducible Schrödinger equations (not necessarily self-adjoint) with periodic coefficients are basically equivalent to some Chebyshev equation.

To understand the structure of the solutions of a second order linear, irreducible and self-adjoint difference equation with periodic coefficients of period p, it is useful to split a solution sequence $v_0, \dots, v_{n+1}$ into the following p subsequences:

$$\begin{aligned} &v_{kp+j}, \quad k = 0, \dots, m, && \text{if} \quad j = 0, \dots, r, \\ &v_{kp+j}, \quad k = 0, \dots, m-1, && \text{if} \quad j = r+1, \dots, p-1. \end{aligned} \tag{8}$$

Therefore, the sequence $v_0,\dots,v_{n+1}$ is splitted into $r+1$ subsequences with $m+1$ entries and $p-1-r$ subsequences with m entries. Of course, when $r=p-1$, all subsequeces have $m+1$ entries.

The structure of the solutions for difference equations with periodic coefficients is contained in the following result:

Lemma 2.3 ([2, Theorem 3.3]) *Let $n\in\mathbb{N}^*$ and $1\le p<n$ and consider $m\in\mathbb{N}^*$ and $0\le r<p$ such that $n+1=mp+r$. Then, there exists $q_p\colon \mathbb{A}_p^{n+2}\times(\mathbb{R}^+)_p^{n+1}\longrightarrow\mathbb{R}$ satisfying that for any $\mathsf{a}\in\mathbb{A}_p^{n+2}$ and $\mathsf{b}\in(\mathbb{R}^+)_p^{n+1}$, $\mathsf{z}\in\mathbb{R}^{n+2}$ is a solution of the homogeneous Schrödinger equation*

$$-b_{k-1}z_{k-1}+a_kz_k-b_kz_{k+1}=0,\ \ k=1,\dots,n.$$

iff for any $j=0,\dots,r$, z_{kp+j} is a solution of the Chebyshev equation

$$-z_{k-1}+2q_p(\mathsf{a},\mathsf{b})z_k-z_{k+1}=0,\ \ k=1,\dots,m$$

and for any $j=r+1,\dots,p-1$, z_{kp+j} is a solution of the Chebyshev equation

$$-z_{k-1}+2q_p(\mathsf{a},\mathsf{b})z_k-z_{k+1}=0,\ \ k=1,\dots,m-1.$$

Therefore, if $\mathsf{z}\in\mathbb{R}^{n+2}$ satisfies that $\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{z})=0$, then

$$z_{kp+j}=z_{p+j}U_{k-1}\big(q_p(\mathsf{a},\mathsf{b})\big)-z_jU_{k-2}\big(q_p(\mathsf{a},\mathsf{b})\big)$$

where $k=0,\dots,m$ when $j=0,\dots,r$ and $k=0,\dots,m-1$ when $j=r+1,\dots,p-1$.

While we know that any solution of a linear difference equation is *implicitly* determined by its two first values, z_0 and z_1 by using the three-terms recurrence, the above result says that the knowledge of the first $2p$ values of any solution of a homogeneous Schrödinger equation with periodic coefficients of period p *explicitly* determines its remainder values. Of course, and as we have previously showed, this result is well-known when $p=1$; that is, for constant coefficients. However, when $p\ge 2$, Lemma 2.3 showed that once this $2p$ values are determined, we get the explicit expression for the solutions. This $2p$ values can be obtained from the two first one; z_0 and z_1 by applying the three-term recurrence

$$\begin{aligned}
z_k&=b_{k-1}^{-1}\big(a_{k-1}z_{k-1}-b_{k-2}z_{k-2}\big), &&k=2,\dots,p, &&\text{when } p\ge 2,\\
z_{p+1}&=b_0^{-1}\big(a_pz_p-b_{p-1}z_{p-1}\big),\\
z_{p+k}&=b_{k-1}^{-1}\big(a_{k-1}z_{p+k-1}-b_{k-2}z_{p+k-2}\big), &&k=2,\dots,p-1, &&\text{when } p\ge 3,
\end{aligned}$$

where we have applied the periodicity of $a's$ and $b's$. Of course, this result will be efficient when $p\ll n$, for small values of p, and for the cases for which the explicit computation of the constant $q_p(\mathsf{a},\mathsf{b})$ is also factible.

The function q_p is named *Floquet function* and its value $q_p(\mathsf{a},\mathsf{b})$ at $\mathsf{a}\in\mathbb{A}_p^{n+2}$ and $\mathsf{b}\in(\mathbb{R}^+)_p^{n+1}$ is called *Floquet constant for the vectors* a *and* b. Therefore, obtaining the fundamental solution for a second order linear difference equation with periodic coefficients depends only on the knowledge of the associated Floquet constant.

As a straightforward consequence of Lemma 2.3, we can obtain the spectral information for Jacobi almost p-Toeplitz matrices. First, it can be proved that $q_p(\mathsf{a}-x\mathsf{e},\mathsf{b})$ is a polinomial with degree equal to p and main coefficient $\frac{1}{2}\Big(\prod_{k=0}^{p-1}b_k\Big)^{-1}$.

For fixed $\mathsf{a} \in \mathbb{A}_p^{n+2}$, we consider its the *periodicity values* $\sigma = a_0 - a_p$ and $\tau = a_{n+1} - a_r$. Therefore, $a_0 = a_p + \sigma$ and $a_{n+1} = a_r + \tau$ and in particular, when $r = 0$ then $a_{n+1} = a_p + \sigma + \tau$. Moreover, $\mathsf{a} \in \mathbb{R}_p^{n+2}$ iff $\sigma = \tau = 0$.

For fixed $\mathsf{a} \in \mathbb{A}_p^{n+2}$ and $\mathsf{b} \in (\mathbb{R}^+)_p^{n+1}$ we define the *associated fundamental polynomials* as $v_k(x)$, $k = 0, \dots, 2p-1$ recurrently given as

$$\begin{aligned}
v_0(x) &= b_0,\\
v_1(x) &= a_0 - x = a_p + \sigma - x,\\
v_k(x) &= b_{k-1}^{-1}\big((a_{k-1} - x)v_{k-1}(x) - b_{k-2}v_{k-2}(x)\big), \qquad k = 2, \dots, p, \qquad \text{if } p \geq 2,\\
v_{p+1}(x) &= b_0^{-1}\big((a_p - x)v_p(x) - b_{p-1}v_{p-1}(x)\big),\\
v_{p+k}(x) &= b_{k-1}^{-1}\big((a_{k-1} - x)v_{p+k-1}(x) - b_{k-2}v_{p+k-2}(x)\big), \quad k = 2, \dots, p-1, \quad \text{if } p \geq 3.
\end{aligned}$$

It is clear that for any $x \in \mathbb{R}$, $v_0(x), v_1(x), \dots, v_{2p-1}(x)$ are the first $2p$ entries of the fundamental solution of the Schrödinger equation $\Delta_{\mathsf{a}-x\mathsf{e},\mathsf{b}}(\mathsf{z}) = 0$ on $\{1, \dots, n\}$.

In addition, for fixed $\mathsf{a} \in \mathbb{A}_p^{n+2}$ and $\mathsf{b} \in (\mathbb{R}^+)_p^{n+1}$ we also define the *coefficient polynomials* as

$$\begin{aligned}
\Phi_0(x) &= (a_p + \sigma + \tau - x)v_p(x) - b_{p-1}v_{p-1}(x),\\
\Psi_0(x) &= b_0(a_p + \sigma + \tau - x) + b_{p-1}\big(v_{2p-1}(x) - 2q(x)v_{p-1}(x)\big),
\end{aligned}$$

and when $p \geq 2$, for any $1 \leq r < p$ as

$$\Phi_r(x) = \tau v_{p+r}(x) + b_r v_{p+r+1}(x) \quad \text{and} \quad \Psi_r(x) = \tau v_r(x) + b_r v_{r+1}(x).$$

Our main result is the following one.

Theorem 2.4 *For any $\mathsf{a} \in \mathbb{A}_p^{n+2}$ and $\mathsf{b} \in (\mathbb{R}^+)_p^{n+1}$ consider $v_0(x), \dots, v_{2p-1}(x)$ and Φ_r, Ψ_r their associated fundamental and coefficient polynomials, respectively. Then, the characteristic polynomial of the Jacobi almost p-Toeplitz matrix $\mathsf{J}(\mathsf{a}, \mathsf{b})$ is*

$$P_{\mathsf{a},\mathsf{b}}(x) = b_0^{-1}\Big(\prod_{j=0}^{r-1} b_j\Big)\Big(\prod_{j=0}^{p-1} b_j^m\Big)\Big[\Phi_r(x)U_{m-1}\big(q_p(\mathsf{a} - x\mathsf{e}, \mathsf{b})\big) - \Psi_r(x)U_{m-2}\big(q_p(\mathsf{a} - x\mathsf{e}, \mathsf{b})\big)\Big].$$

Therefore, $P_{\mathsf{a},\mathsf{b}}(\lambda)$ has $n+2$ different roots and when $P_{\mathsf{a},\mathsf{b}}(\lambda) = 0$, then $\mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda) = \operatorname{span}\{\mathsf{v}(\lambda)\}$, where the entries of $\mathsf{v}(\lambda)$ are given by

$$v_{kp+j}(\lambda) = v_{p+j}(\lambda)U_{k-1}\big(q_p(\mathsf{a} - \lambda\mathsf{e}, \mathsf{b})\big) - v_j(\lambda)U_{k-2}\big(q_p(\mathsf{a} - \lambda\mathsf{e}, \mathsf{b})\big)$$

for any $k = 0, \dots, m$ if $j = 0, \dots, r$ and any $k = 0, \dots, m-1$ if $j = r+1, \dots, p-1$.

The result in Theorem 2.4 is quite general, but is still far from give us manageable expressions for the spectral characteristics of the Jacobi almost-Toeplitz matrices. For instance, finding the roots of the polynomial $P_{\mathsf{a},\mathsf{b}}(x)$ is usually a numerical challenge, since explicit expressions are available only in some special cases. We end this paragraph describing a specific case in which we can go further, although we are still in a fairly general scenario.

When $\mathsf{a} \in \mathbb{A}_p^{n+2}$, $\mathsf{b} \in \mathbb{R}_p^{n+1}$ and $n+1 = mp + r$ with $0 \leq r < p$ we say the pair (a, b) is *β-compatible* if there exists $\beta \in \mathbb{R}$ such that

$$\Phi_r(x) = \big(2q_p(\mathsf{a} - x\mathsf{e}, \mathsf{b}) + \beta\big)\Psi_r(x). \tag{9}$$

When $\beta = 0$ in the above identity, we simply refers it as *compatible*.

Observe that when $r \geq 1$ the coefficients polynomials Φ_r and Ψ_r have degree $p+r+1$ and p respectively, whereas when $r=0$, Φ_0 has degree $p+1$ and Ψ_0 has degree less or equal to $2p-1$. Therefore, when $r=0$, the pair (a,b) only can be β-compatible for some β, when the polynomial $v_{2p-1}(x) - 2q_p(\mathsf{a}-x\mathsf{e},\mathsf{b})v_{p-1}(x)$ has degree less that or equal to 1 and $\Psi_0(x)$ has degree equal to 1.

As we can see below, when the pair (a,b) is compatible, the characteristic polynomial of $\mathsf{J}(\mathsf{a},\mathsf{b})$ is a multiple of the polynomial $U_m\big(q_p(\mathsf{a}-x\mathsf{e},\mathsf{b})\big) + \beta U_{m-1}\big(q_p(\mathsf{a}-x\mathsf{e},\mathsf{b})\big)$, so the eigenvalues of $\mathsf{J}(\mathsf{a},\mathsf{b})$ are related with the zeroes of Chebyshev polynomials. To analyze this situation, we first consider the following result about zeroes of Chebyshev polynomials.

For any $\beta \in \mathbb{R}$ we denote by $z_{m,1}(\beta) > \cdots > z_{m,m}(\beta)$ the ordered m roots of the Chebyshev polynomial $U_m(z) + \beta U_{m-1}(z)$. We know that $z_{m,j}(0) = \cos\big(\frac{j\pi}{m+1}\big)$, $j=1,\ldots,m$, and that for any $\beta \neq 0$, $m-1$ roots strictly interlace $\{z_{m,j}(0)\}_{j=1}^m$ and the other one has modulus greater that $\cos\big(\frac{\pi}{m+1}\big)$.

Lemma 2.5 *Given $m \geq 1$ and $\beta \in \mathbb{R}$, consider $z_{m,1}(\beta) > \cdots > z_{m,m}(\beta)$ the ordered m roots of the Chebyshev polynomial $U_m(z) + \beta U_{m-1}(z)$. Then, the following properties hold:*

1. *If $\beta = 0$, then $z_{m,j}(0) = \cos\big(\theta_j(0)\big)$ where $\theta_{m,j}(0) = \frac{j\pi}{m+1}$, for any $j=1,\ldots,m$.*
2. *If $\beta > 0$, for any $j=1,\ldots,m-1$ there exist $\frac{j\pi}{m+1} < \theta_{m,j}(\beta) < \frac{(j+1)\pi}{m+1}$ such that $z_{m,j}(\beta) = \cos\big(\theta_j(\beta)\big)$. Moreover, there exists $\theta_{m,m}(\beta) > 0$ satisfying that:*
   (a) *When $0 < \beta < 1 + \frac{1}{m}$, $\frac{m\pi}{m+1} < \theta_{m,m}(\beta) < \pi$ and $z_{m,m}(\beta) = \cos\big(\theta_{m,m}(\beta)\big)$.*
   (b) *When $\beta = 1 + \frac{1}{m}$, $\theta_{m,m}(\beta) = \pi$ and $z_{m,m}(\beta) = \cos\big(\theta_{m,m}(\beta)\big)$.*
   (c) *When $\beta > 1 + \frac{1}{m}$, $\theta_{m,m}(\beta) < \log(\beta)$ and $z_{m,m}(\beta) = -\cosh\big(\theta_{m,m}(\beta)\big)$.*
3. *If $\beta < 0$, for any $j=2,\ldots,m$ there exist $\frac{(j-1)\pi}{m+1} < \theta_{m,j}(\beta) < \frac{j\pi}{m+1}$ such that $z_{m,j}(\beta) = \cos\big(\theta_j(\beta)\big)$. Moreover, there exists $\theta_{m,1}(\beta) > 0$ satisfying that:*
   (a) *When $-1 - \frac{1}{m} < \beta < 0$, $0 < \theta_{m,1}(\beta) < \frac{\pi}{m+1}$ and $z_{m,1}(\beta) = \cos\big(\theta_{m,1}(\beta)\big)$.*
   (b) *When $\beta = -1 - \frac{1}{m}$, $\theta_{m,1}(\beta) = 0$ and $z_{m,1}(\beta) = \cos\big(\theta_{m,1}(\beta)\big)$.*
   (c) *When $\beta < -1 - \frac{1}{m}$, $\theta_{m,m}(\beta) < \log(|\beta|)$ and $z_{m,1}(\beta) = \cosh\big(\theta_{m,1}(\beta)\big)$.*

Observe that, with the notation of the above Lemma, for any $j=1,\ldots,m-1$, $\theta_{m,j+1}(-1) = \frac{(2j+1)\pi}{2m+1}$, $\theta_{m,j}(1) = \frac{2j\pi}{2m+1}$, $\theta_{m,1}(-1) = \frac{\pi}{2m+1}$ and $\theta_{m,m}(1) = \frac{2m\pi}{2m+1}$.

Theorem 2.6 *Assume that $\mathsf{a} \in \mathbb{A}_p^{n+2}$ and $\mathsf{b} \in (\mathbb{R}^+)_p^{n+1}$ are β-compatible and let $\sigma,\tau \in \mathbb{R}$ and $v_0(x),\ldots,v_{2p-1}(x)$ be their periodicity coefficients and their associated fundamental polynomials, respectively. Consider also the coefficient polynomial*

$$\Psi_r(x) = \begin{cases} b_0(a_p + \sigma + \tau - x) + b_{p-1}\big(v_{2p-1}(x) - 2q(x)v_{p-1}(x)\big), & \text{when } r = 0, \\ \tau v_r(x) + b_r v_{r+1}(x), & \text{when } 1 \leq r < p, \end{cases}$$

$z_{m,1}(\beta) > \cdots > z_{m,m}(\beta)$ the m roots of the Chebyshev polynomial $U_m(x)+\beta U_{m-1}(x)$ and $\theta_{m,1}(\beta),\ldots,\theta_{m,m}(\beta)$ the arguments given in Lemma 2.5. Then, the following properties hold:

1. *$\Psi_r(x)$ has degree $r+1$ and $r+1$ simple real roots, $\lambda_{mp+1},\ldots,\lambda_{n+2}$.*
2. *For any $j=1,\ldots,m$, the equation $q_p(\mathsf{a}-x\mathsf{e},\mathsf{b}) = z_{m,j}(\beta)$ has p different real solutions $\lambda_{(j-1)p+1},\ldots,\lambda_{jp}$.*
3. *$\{\lambda_k\}_{k=1}^{n+2}$ are the eigenvalues of $\mathsf{J}(\mathsf{a},\mathsf{b})$.*

Moreover, when $|\beta| \leq 1 + \frac{1}{m}$*, then for any* $j = 1, \dots, m$ *and any* $\ell = 1, \dots, p$ *we have that* $\mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda_{(j-1)p+\ell}) = \mathrm{span}\{\mathsf{v}(\lambda_{(j-1)p+\ell})\}$*, where the entries of* $\mathsf{v}(\lambda_{(j-1)p+\ell})$ *are given by*

$$v_{kp+i}(\lambda_{(j-1)p+\ell}) = v_{p+i}(\lambda_{(j-1)p+\ell}) \sin\big(k\theta_{m,j}(\beta)\big) - v_i(\lambda_{(j-1)p+\ell}) \sin\big((k-1)\theta_{m,j}(\beta)\big),$$

for any $k = 0, \dots, m$ *if* $i = 0, \dots, r$ *and any* $k = 0, \dots, m-1$ *if* $i = r+1, \dots, p-1$.

When $|\beta| > 1 + \frac{1}{m}$*, the eigenvectors are the same except* $\mathsf{v}(\lambda_j)$*,* $j = 1, \dots, p$ *for* $\beta < 0$ *or* $\mathsf{v}(\lambda_{(m-1)p+j})$*,* $j = 1, \dots, p$ *for* $\beta > 0$ *for which their entries are given by*

$$\begin{aligned} v_{kp+i}(\lambda_j) &= v_{p+i}(\lambda_j) \sinh\big(k\theta_{m,1}(\beta)\big) - v_i(\lambda_j) \sinh\big((k-1)\theta_{m,j}(\beta)\big), \\ (-1)^k v_{kp+i}(\lambda_{(j-1)p+\ell}) &= v_{p+i}(\lambda_{(j-1)p+\ell}) \sinh\big(k\theta_{m,m}(\beta)\big) \\ &\quad + v_i(\lambda_{(j-1)p+\ell}) \sinh\big((k-1)\theta_{m,m}(\beta)\big). \end{aligned}$$

for any $k = 0, \dots, m$ *if* $i = 0, \dots, r$ *and any* $k = 0, \dots, m-1$ *if* $i = r+1, \dots, p-1$.

In addition to obtain the general expression for the spectral characteristic of Jacobi almost-Toeplitz matrices, in this communication we will also present their expression for the special case when the matrices are bisymmetric; that is, when the vectors a and b are centrosymmetric. In fact, we explicitly obtain the eigenvalues and eigenvectors for low periods.

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