A joint Choquet index from two vectors of weights

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Abstract. The topic of mixing the weighted average means (WAM) and ordered weighted average (OWA) operators has been studied in agonizing detail. It has become one of the most prominent ways to extend OWA operators. Mixed indices using both approaches must use two vectors of weights. Whereas the focus of weights associated with WAM is on the values, the weights associated with OWA focus on order. This work studies how two weights can be combined to produce one single operator sharing traits from both WAMs and OWAs. This operator takes the form of a Choquet integral defined by a 2-additive capacity.

Keywords: Choquet integral, weighted average mean, ordered weighted average operator, capacity.
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1.1 Introduction

Since Yager [28] introduced the ordered weighted average (OWA) operators to aggregate a vector of numbers by a single number, many authors have strived to produce increasingly refined versions, which in many cases involve the utilization of a second vector of weights. This work will also be motivated by the use of two weight vectors. Initially, Torra [23] added the benefits of weighted average means (WAM) to the equal treatment of all values that is intrinsic to OWAs, which are symmetric. This blend produced the successful WOWA operators [4, 25], that have been extended to the continuous case [24] and exported to soft computing models [3]. Other authors followed suit to produce unified operators with these two competing approaches, including [6], [14], and [22].

Choquet integrals encompass both aggregation operators [9]. This theory builds on the concept of capacity [19]. Capacities are known as fuzzy measures too. They are set functions which are monotonic and vanish on the empty set [21]. Informally, they relax the additivity property of measures to monotonicity. WAMs and OWAs arise when the capacity that defines the Choquet integral is either additive or symmetric, respectively. It is no surprise that this class of aggregation operators has supported other attempts to unify WAMs with OWAs. And also, that the Choquet integral has found applications to many areas such as welfare economics [10], social choice [1], soft computing [2], or bibliometrics [26].

In relation with this issue, Yager and Alajlan [29, Sect. 4] suggested the utilization of capacities, plus an aggregation attitudinal function and a quasi-arithmetic generating function, for generalizing importance weighted mean aggregation. The utilization of semi-uninorms in conjunction with two vectors of weights has produced SUOWAs [15, 16, 17]. They consist of Choquet integrals associated with capacities that use semi-uninorms and the values of capacities corresponding to weighted arithmetic means and OWAs. They are different from WOWAs [15], which can nevertheless be expressed as a Choquet integral too. Llamazares [18] insisted on the ability of the Choquet integral to generalize both WAMs and OWAs with the help of capacities designed from two vectors of weights and appropriate functions.

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We show in this work how two vectors of weights can produce a non-additive Choquet integral constructed from a 2-additive capacity. By confining ourselves to this class of capacities, we eliminate the requirement for additional external elements as those reported above.

Our construction of a set function from the two vectors is explicit. Whenever this set function becomes the Möbius inverse of a (2-additive) capacity, it defines a discrete Choquet integral that we call the joint Choquet index defined from two vectors of weights. The set functions that are Möbius inverse of a capacity are completely identified [8]. With this characterization we set forth sufficient conditions for the two vectors of weights to return a joint Choquet index. Then we explore the properties of this construction and its limitations. We find that under certain circumstances, the index behaves as a sum of WAM and (modified) OWA on a restricted domain of vectors. Examples illustrate the utilization of this technique.

In conclusion, we can therefore relate the possibility of generalizing WAMs through a second vector of weights with the Choquet integrals defined from 2-additive capacities. An advantage of our index with respect to an unrestricted Choquet integral is that it needs few parameters, and at the same time, it leverages the important Choquet aggregation operator with the advantages of two independent sets of weights.

1.2 Background

We shall assume that the set $X = \{1, \ldots, n\}$ represents a list of indicators. The goal is to produce evaluations of numerical values indexed by X, with certain properties related to those of the next two aggregation procedures:

Definition. Let $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$ be a weighting vector such that $\sum_{i=1}^n w_i = 1$. The **weighted average mean** (WAM) associated with $\mathbf{w}$ is the following function $WAM^{\mathbf{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$WAM^{\mathbf{w}}(a_1, \ldots, a_n) = \sum_{i=1}^n w_i\, a_i, \ \text{ for every } \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n. \tag{1.1}$$

Definition (Yager [28]). Let $\mathbf{v} = (v_1, \ldots, v_n) \in [0,1]^n$ be a weighting vector such that $\sum_{i=1}^n v_i = 1$. The **ordered weighted averaging** (OWA) operator associated with $\mathbf{v}$ is the following function $O^{\mathbf{v}} : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$O^{\mathbf{v}}(a_1, \ldots, a_n) = \sum_{i=1}^n v_i\, b_i, \ \text{ for every } \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n. \tag{1.2}$$

In this expression, b_i is the i-th largest element in the collection of (possibly repeated) values $\{a_1, \ldots, a_n\}$. This means that when $\mathbf{a}_{\searrow} = (a_{[1]}, \ldots, a_{[n]})$ is a non-increasing permutation of $\mathbf{a}$, then $b_i = a_{[i]}$.

Very important for our exercise is the next concept:

Definition. A discrete **capacity** on X is a set function $\mu : 2^X \longrightarrow [0,1]$ that is monotonic (i.e., $\mu(A) \leqslant \mu(B)$ whenever $A \subseteq B \subseteq X$) and satisfies $\mu(\varnothing) = 0$, $\mu(X) = 1$.

Although the term 'capacity' was coined by Choquet [9], Sugeno [21] independently defined the same notion under the name **fuzzy measure**.

A (discrete) capacity μ is **additive** when in case that $A, B \subseteq X$ are disjoint, then the equality $\mu(A \cup B) = \mu(A) + \mu(B)$ holds true. Additive capacities are measures. Besides, μ is **symmetric** when $\mu(A) = \mu(B)$ provided that $A, B \subseteq X$ and $|A| = |B|$.

It has been argued that "[p]erhaps the most successful subfamily [of capacities] is the subfamily of k-additive measures, and inside this subfamily, the most appealing case is the case of 2-additive measures" (Miranda and García-Segador [20]). The main reason is computational tractability. To define a capacity on X we need $2^n - 2$ numbers that in addition, must satisfy the mandatory monotonicity constraints. Many researchers have produced simplified but flexible expressions of a capacity, and to this purpose the k-additive capacities have proven to be especially reliable. With

$\sum_{i=1}^{k} C_n^i$ evaluations one can to define a k-additive capacity [13]. This concept can be given a compact definition with the help of the Möbius inverse [11], an operator that returns the capacity that produced it through the Zeta transform. Let us define the operator that we need in this work:

Definition [12, Def. 2.30]. The **Möbius inverse** of the discrete capacity μ is the unique solution to the set of equations

$$\mu(X') = \sum_{A \subseteq X'} m^{\mu}(A), \text{ for each } X' \subseteq X. \tag{1.3}$$

This solution is given by the next set function m^{μ} on X:

$$m^{\mu}(X') = \sum_{A \subseteq X'} (-1)^{|X' \setminus A|} \mu(A), \text{ for each } X' \subseteq X.$$

Equation (1.3) defines the **Zeta transform** of m^{μ}.

Additive capacities are characterized by having Möbius inverses whose evaluations on sets with more than one element are null [12, Th. 2.33]. In a similar vein, the next extension of additivity has been defined:

Definition (Grabisch [11]). A discrete capacity μ on X is **k-additive** when a subset of X with cardinality k exists whose evaluation by m^{μ} is not zero; and for all $X' \subseteq X$ with $k+1$ elements or more, the evaluation of X' by m^{μ} is zero.

Additive capacities coincide with 1-additive capacities by the aforementioned [12, Th 2.33].

Note that any 2-additive capacity is totally determined by the values that it attains on singletons and doubletons, since we have

$$\mu(X') = \sum_{\substack{i \neq j \\ i,j \in X'}} \mu(\{i,j\}) - (|X'| - 2) \sum_{i \in X'} \mu(\{i\}), \text{ for each } X' \subseteq X.$$

Some simple computations demonstrate that $\frac{n\,(n+1)}{2}$ values suffice to define a 2-additive capacity on a set with n elements.

When m^{μ} is the Möbius inverse of a 2-additive capacity μ,

$$\sum_{\substack{i \neq j \\ i,j \in X}} m^{\mu}(\{i,j\}) = 1 - \sum_{i \in X} \mu(\{i\})$$

Which mappings with the right structure $m : 2^X \longrightarrow \mathbb{R}$ behave as a 2-additive capacity on X? The next result gives a precise answer:

Theorem (Chateauneuf and Jaffray [8]). The set-valued function $m : 2^X \longrightarrow \mathbb{R}$ is the Möbius inverse of a capacity μ on X if and only if the next two conditions hold true:

1. $m(\varnothing) = 0$ and $\sum_{X' \subseteq X} m(X') = 1$.
2. For each $X' \subseteq X$ and $j \in X'$: $\sum_{j \in A \subseteq X'} m(A) \geqslant 0$.

Since we need to apply this test to secure 2-additive capacities, it is worth stating that the second restriction becomes in this case:

2. For each $X' \subseteq X$ and $j \in X'$: $\sum_{i \in X'} m(\{i,j\}) \geqslant 0$.

The latter condition in particular imposes $m(\{i,j\}) + \mu(\{j\}) \geqslant 0$ for all i, j (which is the application of the property to the case $X' = \{i,j\}$ and j). The restriction enforces monotonicity of μ defined from m by the Zeta transform, whereas (1) assures the boundary conditions (i.e., $\mu(\varnothing) = 0$ and $\mu(X) = 1$).

Capacities define Choquet integrals as follows:

Definition (Choquet [9]). The discrete Choquet integral with respect to the capacity μ is $C^{\mu} : \mathbb{R}_+^n \longrightarrow \mathbb{R}$ where $C^{\mu}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \left[a_{(j)} - a_{(j-1)}\right] \mu(L_j)$, and $\mathbf{a}_{\nearrow} = (a_{(1)}, \ldots, a_{(n)})$ is a non-decreasing permutation of $\mathbf{a} = (a_1, \dots, a_n)$, we let $a_{(0)} = 0$, and $L_j = \{(j), \dots, (n)\}$ is the set of indices of the $n - j + 1$ largest components of $\mathbf{a}$.

It is also remarkable that there is an explicit correspondence between WAMs (resp., OWAs) and Choquet integrals defined from additive (resp., symmetric) capacities [12, Th. 4.63] also [17, Rem. 3]. Especially for our needs, the WAM defined from $\mathbf{w} = (w_1, \dots, w_n) \in [0,1]^n$ on $\mathbb{R}^n_+$ is the Choquet integral with respect to the additive capacity $\mu^{\mathbf{w}}$ such that $\mu^{\mathbf{w}}(\{i\}) = w_i$ for all $i \in X$.

1.3 Results

Hereafter we let $\mathbf{w} = (w_1, \dots, w_n) \in [0,1]^n$ and $\mathbf{v} = (v_1, \dots, v_n) \in [0,1]^n$ be weighting vectors such that $\sum_{i=1}^n w_i = \sum_{i=1}^n v_i = 1$. Several attempts have been made to combine the WAM and OWA respectively defined from these vectors. For example:

1. The OWAWA (ordered weighted averaging-weighted average) operator [22] simply uses a convex combination of both operators: with a new parameter $\alpha \in [0,1]$,

$$OWAWA(\mathbf{a}) = \alpha WAM^{\mathbf{w}}(\mathbf{a}) + (1-\alpha)O^{\mathbf{v}}(\mathbf{a}), \text{ for every } \mathbf{a} \in \mathbb{R}^n. \tag{1.4}$$

2. The SDOWA (standard deviation OWA) operator [7] uses the same formula, however the authors insist that the α weighting parameter should be directly defined from $\mathbf{w}$ and $\mathbf{v}$ by their standard deviations: $\alpha = \frac{sd(\mathbf{w})}{sd(\mathbf{w})+sd(\mathbf{v})}$. This choice guarantees the coincidence with $O^{\mathbf{v}}$ when $\mathbf{w}$ is constant, and coincidence with $WAM^{\mathbf{w}}$ when $\mathbf{v}$ is constant.

3. The HWA (hybrid weighted average) operator [27] composes the OWA defined from $\mathbf{w}$ with a function directly defined from the other vector $\mathbf{v}$: the application on $\mathbf{a} \in \mathbb{R}^n$ is

$$HWA(\mathbf{a}) = OWA^{\mathbf{v}}(n\mathbf{w}\mathbf{a}) = OWA^{\mathbf{v}}(nw_1a_1, \dots, nw_na_n).$$

4. The JWA (joint weighted average) operator [6] allows the weights to interact. As in the case of SDOWA, it does not introduce further parameters. To apply JWA on $\mathbf{a} \in \mathbb{R}^n$, one writes $\mathbf{a}_{\searrow} = (a_{[1]}, \dots, a_{[n]})$ is a non-increasing permutation of $\mathbf{a}$, then produces the similarly ordered vector of weights $\mathbf{w}_{[\mathbf{a}]} = (w_{[1]}, \dots, w_{[n]})$, and applies the formula

$$JWA(\mathbf{a}) = (\mathbf{w}_{[\mathbf{a}]} \oplus \mathbf{v})\mathbf{a}_{\searrow} \tag{1.5}$$

where $\oplus$ is defined as in compositional geometry, i.e.,

$$\mathbf{x} \oplus \mathbf{y} = \left(\frac{x_1 y_1}{\sum_{i=1}^n x_i y_i}, \dots, \frac{x_n y_n}{\sum_{i=1}^n x_i y_i} \right) \text{ when } \mathbf{x} = (x_1, \dots, x_n),\ \mathbf{y} = (y_1, \dots, y_n).$$

We do not recall WOWAs here. Suffice to say that this concept uses an interpolation function, in addition to two vectors of weights. And also that the result depends on the interpolation function, which is not uniquely determined.

1.3.1 The joint Choquet index: definition and sufficient conditions

We proceed to study the properties of a proposal that requires the introduction of neither further parameters nor external functions. By doing so we avoid the problem of eliciting the corresponding additional factor.

Definition. Let $P = \frac{\sum_{i \neq j}(w_i w_j - v_i v_j)}{\binom{n}{2}}$. Define the set function $m^{\mathbf{w},\mathbf{v}}$ on X as follows:

$$m^{\mathbf{w},\mathbf{v}}(X') = \begin{cases} 0, & \text{when } X' = \varnothing \text{ or } X' \subseteq X, |X'| > 2 \\ w_i, & \text{when } X' = \{i\} \text{ for some } i \in X, \\ w_i w_j - v_i v_j - P, & \text{when } X' = \{i,j\} \text{ for some } i,j \in X,\ i \neq j. \end{cases}$$

Definition. Whenever $m^{\mathbf{w},\mathbf{v}}$ defines a capacity $\mu^{\mathbf{w},\mathbf{v}}$, we say that the Choquet integral defined by $\mu^{\mathbf{w},\mathbf{v}}$ is the **joint Choquet index associated with w** and **v**.

Note that by definition, the joint Choquet index defined above shares the following property with the WAM defined from $\mathbf{w} = (w_1, \ldots, w_n)$: both are Choquet integrals respectively defined by capacities $\mu^{\mathbf{w},\mathbf{w}}$ and $\mu^{\mathbf{w}}$ such that $\mu^{\mathbf{w},\mathbf{w}}(\{i\}) = \mu^{\mathbf{w}}(\{i\}) = w_i$ for all $i \in X$. The next section expands on the properties of the joint Choquet index associated with identical vectors.

Using Chateauneuf and Jaffray's theorem stated above, we can prove that $m^{\mathbf{w},\mathbf{v}}$ is the Möbius inverse of a capacity $\mu^{\mathbf{w},\mathbf{v}}$ under a set of (at most $n+1$) sufficient conditions that can be easily checked.

Proposition 1. Suppose that
(i) $P \leqslant 0$ (i.e., $\sum_{i\neq j} w_i w_j \leqslant \sum_{i\neq j} v_i v_j$), and
(ii) for all $j \in X$ such that $w_j < v_j$, it is the case that $v_j^2 \geqslant v_j - w_j - w_j \cdot \min_{i\neq j} w_i$.
Then $m^{\mathbf{w},\mathbf{v}}$ is the Möbius inverse of a capacity $\mu^{\mathbf{w},\mathbf{v}}$.

Proof. The definition of $m^{\mathbf{w},\mathbf{v}}$ guarantees that property 1 in the characterization holds true.

To prove property 2, we fix $X' \subseteq X$ and $j \in X'$. We need to check $\sum_{i\in X'} m^{\mathbf{w},\mathbf{v}}(\{i,j\}) \geqslant 0$. Direct computations show that $\sum_{i\in X'} m^{\mathbf{w},\mathbf{v}}(\{i,j\}) = m^{\mathbf{w},\mathbf{v}}(\{j\}) + \sum_{j\neq i\in X'} m^{\mathbf{w},\mathbf{v}}(\{i,j\}) \geqslant 0$ is equivalent to $w_j + \sum_{j\neq i\in X'}(w_i w_j - v_i v_j) \geqslant P(|X'|-1)$, or $w_j(1+\sum_{j\neq i\in X'} w_i) - v_j \sum_{j\neq i\in X'} v_i \geqslant P(|X'|-1)$.

Suppose first $w_j - v_j \geqslant 0$. Then $w_j(1+\sum_{j\neq i\in X'} w_i) - v_j(\sum_{j\neq i\in X'} v_i) \geqslant w_j - v_j \geqslant 0 \geqslant P(|X'|-1)$.

Now suppose $w_j - v_j < 0$. Using $\sum_{j\neq i\in X'} v_i \leqslant \sum_{j\neq i\in X} v_i = 1 - v_j$, we have

$$w_j\Big(1 + \sum_{j\neq i\in X'} w_i\Big) - v_j\Big(\sum_{j\neq i\in X'} v_i\Big) \geqslant w_j\Big(1 + \sum_{j\neq i\in X'} w_i\Big) - v_j(1-v_j) \geqslant w_j + w_j \cdot \min_{i\neq j} w_i - v_j + v_j^2$$

and assumption (ii) guarantees the conclusion $w_j + w_j \cdot \min_{i\neq j} w_i - v_j + v_j^2 \geqslant 0 \geqslant P(|X'|-1)$. □

In the next sections we show that neither condition (i) nor the set of conditions (ii) are necessary.

1.3.2 Properties of the joint Choquet index

We proceed to state several properties of the joint Choquet index. These properties add to the standard properties derived from the fact that it is defined as a Choquet integral (e.g., compensativeness, monotonicity, idempotency, or positive homogeneity of degree 1). They help us find similarities with WAMs and their combinations with OWAs.

1. For each $i \in X$, $\mu^{\mathbf{w},\mathbf{v}}(\{i\}) = m^{\mathbf{w},\mathbf{v}}(\{i\}) = w_i$. This assignment is mandatory for capacities producing additive Choquet integrals that coincide with the WAM defined from $\mathbf{w}$ [12, Eq. (4.77)].

2. In the particular case where $\mathbf{w} = \mathbf{v}$, the joint Choquet index associated with $\mathbf{w}$ and $\mathbf{v}$ is well defined and it coincides with $WAM^{\mathbf{w}} = WAM^{\mathbf{v}}$. Note that $P = 0$ and (ii) above holds vacuously. In fact, $m^{\mathbf{w},\mathbf{v}}(\{i,j\}) = 0$ when $i \neq j$, proving additivity.

3. More particularly, when $\mathbf{w} = \mathbf{v} = (\frac{1}{n}, \ldots, \frac{1}{n})$, the joint Choquet index associated with $\mathbf{w}$ and $\mathbf{v}$ coincides with the simple average mean (i.e., the arithmetic average). And this is the only OWA operator defined as a joint Choquet index associated with two vectors.

4. When $\mathbf{v}$ is such that $v_i = 1$ for some i, then $\mu^{\mathbf{w},\mathbf{v}}$ is additive if and only if it is symmetric. And in this case, $\mathbf{w} = (\frac{1}{n}, \ldots, \frac{1}{n})$, therefore the joint Choquet index associated with these vectors is the arithmetic average too.

5. When $\mathbf{w}$ is such that $w_i = 1$ for some i, then the only capacity $\mu^{\mathbf{w},\mathbf{v}}$ that is additive is the Dirac measure centered at i. It is defined by

$$\mu^{\mathbf{w},\mathbf{v}}(X') = \begin{cases} 1, & \text{when } i \in X', \\ 0, & \text{otherwise} \end{cases} \quad \text{(cf., [12, Sect. 2.2])}$$

exactly through the vector of weights $\mathbf{v} = (\frac{1}{n}, \ldots, \frac{1}{n})$.

In this case, [12, Cor. 4.64 (iii)] assures that the joint Choquet index associated with these vectors is the projection on the i-th coordinate. Of course, it is the WAM associated with $\mathbf{w}$.

Importantly, (ii) in Proposition 1 is contradicted when $n > 2$, thus proving that the set of requirements (ii) is not necessary.

6. Suppose $\mathbf{w} = (\frac{1}{n}, \ldots, \frac{1}{n})$, and $v_i + v_j = 1$ for some $i \neq j$. Then the joint Choquet index associated with $\mathbf{w}$ and $\mathbf{v}$ is well defined, and at every $\mathbf{a} \in \mathbb{R}^n_+$ with $a_i \cdot a_j = 0$, it attains the value

$$WAM^{\mathbf{w}}(\mathbf{a}) + v_i \cdot (1 - v_i) \cdot OWA^{\mathbf{n}} \quad \text{with} \quad \mathbf{n} = \Big(\frac{n-1}{\binom{n}{2}}, \frac{n-2}{\binom{n}{2}}, \ldots, \frac{1}{\binom{n}{2}}, 0\Big).$$

7. Suppose $\mathbf{v} = (\frac{1}{n}, \ldots, \frac{1}{n})$, and $w_i + w_j = 1$ for some $i \neq j$. Then the joint Choquet index associated with $\mathbf{w}$ and $\mathbf{v}$ is well defined, and at every $\mathbf{a} \in \mathbb{R}^n_+$ with $a_i \cdot a_j = 0$, it attains the value

$$WAM^{\mathbf{w}}(\mathbf{a}) - w_i \cdot (1 - w_i) \cdot OWA^{\mathbf{n}} \quad \text{with} \quad \mathbf{n} = \Big(\frac{n-1}{\binom{n}{2}}, \frac{n-2}{\binom{n}{2}}, \ldots, \frac{1}{\binom{n}{2}}, 0\Big).$$

The proofs of the last two properties exploit the formula that produces the Choquet integral when the capacity is 2-additive, to wit:

$$C^{\mu}(\mathbf{a}) = \sum_{i=1}^{n} m^{\mu}(\{i\}) \cdot a_i + \sum_{\{i,j\} \subseteq N} m^{\mu}(\{i, j\}) \cdot \min\{a_i, a_j\}$$

for each $\mathbf{a} = (a_1, \ldots, a_n)$. This formula derives from the general expression of a Choquet integral from the Möbius inverse of the capacity that defines it [5, Sect 2.6].

1.3.3 Numerical examples

The next example illustrates the application of the set of sufficient conditions provided in Proposition 1 in a case with $n = 4$.

Example 1. Suppose $\mathbf{w} = (0.4, 0.2, 0.15, 0.25)$ and $\mathbf{v} = (0.3, 0.3, 0.2, 0.2)$. Then $\sum_{i \neq j} w_i w_j = 0.3575 < 0.37 = \sum_{i \neq j} v_i v_j$, which accounts for (i) in our set of sufficient conditions.

Also, to check (ii), note that $w_1 > v_1$ and $w_4 > v_4$. We only need to do simple computations for $i = 2$ and $i = 3$. And indeed,

$v_2^2 \geqslant v_2 - w_2 - w_2 \cdot \min_{j \neq 2} w_j$ because $0.3^2 \geqslant 0.3 - 0.2 - 0.2 \cdot 0.15$, or $0.09 \geqslant 0.07$, and

$v_3^2 \geqslant v_3 - w_3 - w_3 \cdot \min_{j \neq 3} w_j$ because $0.2^2 \geqslant 0.2 - 0.15 - 0.15 \cdot 0.2$, or $0.04 \geqslant 0.2$.

Therefore the next standard presentation defines $m^{\mathbf{w},\mathbf{v}}$, which must be the Möbius inverse of a capacity (we use four-digit approximations):

			0			
	0	0		0	0	
-0.0067	-0.0167	0.0633		-0.0367	0.0033	-0.0067
	0.4	0.2		0.15	0.25	
			0			

Now the Zeta transform produces the 2-additive capacity $\mu^{\mathbf{w},\mathbf{v}}$ from this expression. It can be represented as follows:

			1			
	0.64	0.96		0.84	0.56	
0.5933	0.4833	0.7633		0.2633	0.5033	0.3933
	0.4	0.2		0.15	0.25	
			0			

Remark. These arrangements replicate the Hasse diagram of the inclusion relation defined on the parts of X.

One pending issue is whether condition (i) in Proposition 1 is necessary. The next example proves that this is not the case.

Example 2. With $n = 3$, suppose $\mathbf{w} = (0.4, 0.35, 0.25)$ and $\mathbf{v} = (0.5, 0.3, 0.2)$. Then $\sum_{i \neq j} w_i w_j = 0.3275 > 0.31 = \sum_{i \neq j} v_i v_j$, and $P = \frac{0.0175}{3} \approx 0.00583 > 0$. We can routinely define $m^{\mathbf{w},\mathbf{v}}$ as follows:

	0	
-0.01583	-0.00583	0.02167
0.4	0.35	0.25
	0	

Now the Zeta transform produces a 2-additive capacity $\mu^{\mathbf{w},\mathbf{v}}$, namely:

	1	
0.734167	0.644167	0.621667
0.4	0.35	0.25
	0	

1.4 Conclusions

We have explored the properties of a way to combine two vectors of weights with the help of the Choquet integral. There are similarities of the joint Choquet index that has been defined, with both weighted average means and (to a lesser degree) ordered weighted average operators. Although joint Choquet indices generalize weighted average means, we cannot extend OWAs in general: the average mean is the only OWA operator produced by a joint Choquet index defined as in this work.

Further research might disclose the exact conditions under which the joint Choquet index is well defined. Interpretations of the roles of the two vectors of indices are also worth investigating to fully grasp the abilities of this index.

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