# Practical approximation of Lyapunov exponents of population models

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## Abstract

Lyapunov exponents (LEs) measure the average exponential asymptotic behavior of solutions in the long time. They can be used to study the asymptotic stability of solutions, to detect the insurgence of chaotic dynamics, to measure the effect of perturbations and to estimate the entropy of the system and the dimension of attractors.

The theory and computation of LEs of ordinary differential equations (ODEs) is a widely studied problem. A commonly used family of approximation methods is based on the QR factorization; we recall, in particular, the discrete QR (DQR) method [6], which is applied to the linearization of the ODE around a reference solution in the attractor.

Evolution equations are a principal tool in modelling population dynamics. Besides ODEs, evolution equations include delay equations, of both differential and renewal type, and partial differential equations (PDEs). The state spaces for delay equations and PDEs are infinite-dimensional function spaces, as opposed to the finite-dimensional vector spaces of ODEs: for the former it is thus necessary to resort to discretization techniques.

In the specific case of the computation of LEs, one possible approach is to discretize the evolution equation via collocation and to apply the DQR method to the resulting system of ODEs. This approach has been used for delay differential equations (DDEs) in [2] and for renewal equations (REs) in [3] using a pseudospectral collocation of the state, and for age-structured population models formulated as PDEs in [1] using a semidiscretization scheme. The aim is at providing a practical method for the approximation of the LEs: in fact, the resulting method is experimentally effective and exhibits the expected convergence properties, but a proof of convergence is still lacking.

Another approach consists in posing the linear(ized) equation in a suitable Hilbert space, discretizing the associated family of evolution operators and adapting the DQR method to the resulting finite-dimensional approximations of the operators. Moreover, for the proof of convergence the DQR method needs to be lifted to infinite dimension. This second approach has been proposed in [5] for DDEs and in [4] for REs, along with the convergence proof, and is currently under investigation for age-structured population models.

In this contribution we present the first approach for REs [3] and for age-structured population models [1], showing its effectiveness and convergence properties via some examples.

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## References

[1] Angulo, Ó., Liessi, D., Practical computation of Lyapunov exponents of age-structured population models, in preparation.

[2] Breda, D., Della Schiava, S., Pseudospectral reduction to compute Lyapunov exponents of delay differential equations, *Discrete Contin. Dyn. Syst. Ser. B* 23(7):2727-2741, 2018, DOI: 10.3934/dcdsb.2018092.

[3] Breda, D., Liessi, D., A practical approach to computing Lyapunov exponents of renewal and delay equations, *Math. Biosci. Eng.* 21(1):1249-1269, 2024, DOI: 10.3934/mbe.2024053.

[4] Breda, D., Liessi, D., Lyapunov exponents of renewal equations: numerical approximation and convergence analysis, submitted, arXiv: 2404.11191 [math.NA, math.DS].

[5] Breda, D., Van Vleck, E., Approximating Lyapunov exponents and Sacker–Sell spectrum for retarded functional differential equations, *Numer. Math.* 126(2):225-257, 2014, DOI: 10.1007/s00211-013-0565-1.

[6] Dieci, L., Jolly, M. S., Van Vleck, E. S., Numerical techniques for approximating Lyapunov exponents and their implementation, *J. Comput. Nonlinear Dynam.*, 6(1):011003, 2011, DOI: 10.1115/1.4002088.