Stability Analysis of a New Fourth-Order Optimal Iterative Scheme Jarratt-Type

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1 Introduction

Mathematics has become a fundamental tool for addressing applied-world problems in various areas such as physics, engineering, or biology.

For example, the quadratic equation $h_0 + v_0 t - \frac{1}{2}gt^2 = 0$ helps us determine the moment when an object launched vertically upwards returns to the ground. This problem can be solved using the quadratic formula $t = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where $a = -\frac{1}{2}g$, $b = v_0$, and $c = h_0$.

However, not all situations in the applied world that are solved with a mathematical model can be reduced to an equation with an existing solution formula. For instance, if the mathematical model associated with a problem reduces to a fifth-degree polynomial equation, then it is not possible to find a formula that determines these solutions using only radicals and basic arithmetic operations. This was demonstrated by the mathematician Niels Henrik Abel in 1824. Hence arises the necessity of fixed-point iterative schemes, as they allow us to determine the solutions of equations without the need for a formula. Due to advancements in computer science, these numerical methods have turned into a very popular and highly utilized (see, for example, [1, 2, 3, 4, 5, 6]). They have become a practical and efficient tool due to the computational power and numerical precision of computers, as well as the development of software such as Matlab and Python, specialized for numerical computation and iterative methods.

Given their versatility, we can use iterative schemes to solve various types of nonlinear equations. However, there are factors that determine whether a method performs better or worse than another

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in approximating a solution of an equation. One of these factors is its dependence on the initial estimate x_0. If the function whose zeros we are approximating is not very smooth or if it has multiple possible solutions, it could slow down convergence or converge to a fixed point that is not a solution of the equation, posing a problem of local convergence and numerical sensitivity. For this reason, there are researchers in the field of numerical analysis devoted to the convergence and stability of new families of iterative methods (see, for example, [7, 8, 9, 10, 11]).

In this research, employing the weight function technique and as a generalization of the method by Jarratt et al. in [13], in Section 2, we establish the form of the weight function of the iterative scheme for constructing our new family and impose necessary and sufficient conditions on this weight function, which ensure that each member of the new family is an optimal method, along with some characteristics of the weight function.

In Section 3, utilizing discrete complex dynamics, we analyze the stability over quadratic polynomials of each member of the family for the parameters that specialize the method contained within a certain determined range.

2 Construction of a new parametric family of Jarratt-type iterative methods

In this section we introduce a family of fourth-order iterative methods whose members are optimal according to the conjecture of Kung and Traub et al. in [12]. This family includes members stable over quadratic polynomials and competitive compared to other methods in the literature in terms of considerable stability.

Jarratt's method, constructed in [13] has, as iterative expression,

$$\begin{aligned} y_i &= x_i - \frac{2}{3}\frac{f(x_i)}{f'(x_i)}, \\ x_{i+1} &= x_i - \frac{1}{2}\left[\frac{3f'(y_i)+f'(x_i)}{3f'(y_i)-f'(x_i)}\right]\frac{f(x_i)}{f'(x_i)},\ i=0,1,\ldots, \end{aligned} \tag{1}$$

This iterative scheme can be observed to have the following structure.

$$\begin{aligned} y_i &= x_i - \frac{2}{3}\frac{f(x_i)}{f'(x_i)}, \\ x_{i+1} &= x_i - H[\mu(x_i)]\frac{f(x_i)}{f'(x_i)},\ i=0,1,\ldots, \end{aligned} \tag{2}$$

where $H[\mu(x_i)] = \frac{1}{2}\left[\frac{3f'(y_i)+f'(x_i)}{3f'(y_i)-f'(x_i)}\right]$, if we divide both the numerator and denominator by $f'(y_i)$ then we have that $H[\mu(x_i)] = \left[\frac{3+\frac{f'(x_i)}{f'(y_i)}}{6-2\frac{f'(x_i)}{f'(y_i)}}\right]$, by making $\mu(x_i) = \frac{f'(x_i)}{f'(y_i)}$ we have that Jarratt's method (1) belongs to a family of iterative schemes of the form (2), where $H[\mu(x)]$ is a weight function with the following structure

$$H[\mu(x)] = \frac{s\mu(x)^k + q\mu(x)^g}{n\mu(x)^l + m\mu(x)^w}.$$

If we consider k, g, l, and w as non-negative integers, then the following result establishes the necessary and sufficient conditions that $H[\mu(x)]$ must satisfy for the members of the iterative scheme family

$$\begin{aligned} y_i &= x_i - \frac{2}{3}\frac{f(x_i)}{f'(x_i)}, \\ x_{i+1} &= x_i - \left[\frac{s\mu(x_i)^k + q\mu(x_i)^g}{n\mu(x_i)^l + m\mu(x_i)^w}\right]\frac{f(x_i)}{f'(x_i)},\ i = 0, 1, \ldots, \end{aligned} \tag{3}$$

be optimal fourth-order methods.

Theorem 2.1 *If $\bar{x} \in I$ is a simple root of a sufficiently differentiable function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, then the method (3) has a convergence order of 4 if and only if,*

$$g \neq k,\ l \neq w,\ -4l(g+k) + g(4k-3) - 3k + 4l^2 + 6l + 6 \neq 0$$

and

$$H_{(k,g,l,w)}[\mu(x)] = \begin{cases} \dfrac{s_1(g,0,w) + q_1(0,0,w)\mu(x)^{g-\lambda}}{n_1(0,g,w) + m_1(0,g,0)\mu(x)^{w-\lambda}} & \text{if} \quad \lambda = l \ \text{and}\ \delta_1 = \beta_1 \\[2ex] \dfrac{s_1(g,0,w)\mu(x)^{k-\lambda} + q_1(k,0,w)\mu(x)^{g-\lambda}}{n_1(k,g,w) + m_1(k,g,0)\mu(x)^{w-\lambda}} & \text{if} \quad \lambda = l \ \text{and}\ \delta_1 \neq \beta_1 \\[2ex] \dfrac{s_1(g,l,w) + q_1(0,l,w)\mu(x)^{g-\lambda}}{n_1(0,g,w)\mu(x)^{l-\lambda} + m_1(0,g,l)\mu(x)^{w-\lambda}} & \text{if} \quad \lambda = k \ \text{and}\ \delta_1 \neq \beta_1 \end{cases}$$

where

$$s_1(g,l,w) = -(l-w)\left(8g^2 - 4g(2l+2w+3) + l(8w+6) + 6w - 3\right),$$

$$q_1(k,l,w) = (l-w)\left(8k^2 - 4k(2l+2w+3) + l(8w+6) + 6w - 3\right),$$

$$n_1(k,g,w) = 2(g-k)(g(-4k+4w+3) + k(4w+3) - 2w(2w+3) - 6),$$

$$m_1(k,g,l) = -2(g-k)(g(-4k+4l+3) + k(4l+3) - 2l(2l+3) - 6),$$

$$\lambda = \min\{\delta_1, \beta_1\},\ \delta_1 = \min\{k,g\},\ \beta_1 = \min\{l,w\},$$

and the error equation is given by the expression

$$e_{i+1} = \left(c_2^3(G_1 - G_2 + G_3 + G_4) - c_3c_2 + \frac{c_4}{9}\right)e_i^4,$$

being

$$c_j = \frac{1}{j!}\frac{f^{(j)}(\bar{x})}{f'(\bar{x})},\ \text{for } j = 2, 3, \ldots,\ \text{and} \quad e_i = x_i - \bar{x},$$

$$G_1 = \frac{8g^2\left(8k^2 - 4k(2l + 2w + 3) + l(8w + 6) + 6w - 3\right)}{81(2g + 2k - 2l - 2w - 3)},$$

$$G_2 = \frac{2g\left(16k^2(2l + 2w + 3) - 4k\left(8l^2 + 8l(2w + 3) + 8w^2 + 24w + 33\right) + 8l^2(4w + 3) + N\right)}{81(2g + 2k - 2l - 2w - 3)},$$

$$G_3 = \frac{8k^2(l(8w + 6) + 6w - 3) - 2k\left(8l^2(4w + 3) + N\right)}{81(2g + 2k - 2l - 2w - 3)},$$

$$G_4 = \frac{64l^2w^2 + 96l^2w + 96l^2 + 96lw^2 + 24lw - 210l + 96w^2 - 210w - 207}{81(2g + 2k - 2l - 2w - 3)},$$

$$N = l\left(32w^2 + 96w + 54\right) + 3\left(8w^2 + 18w - 35\right).$$

3 Dynamical Planes

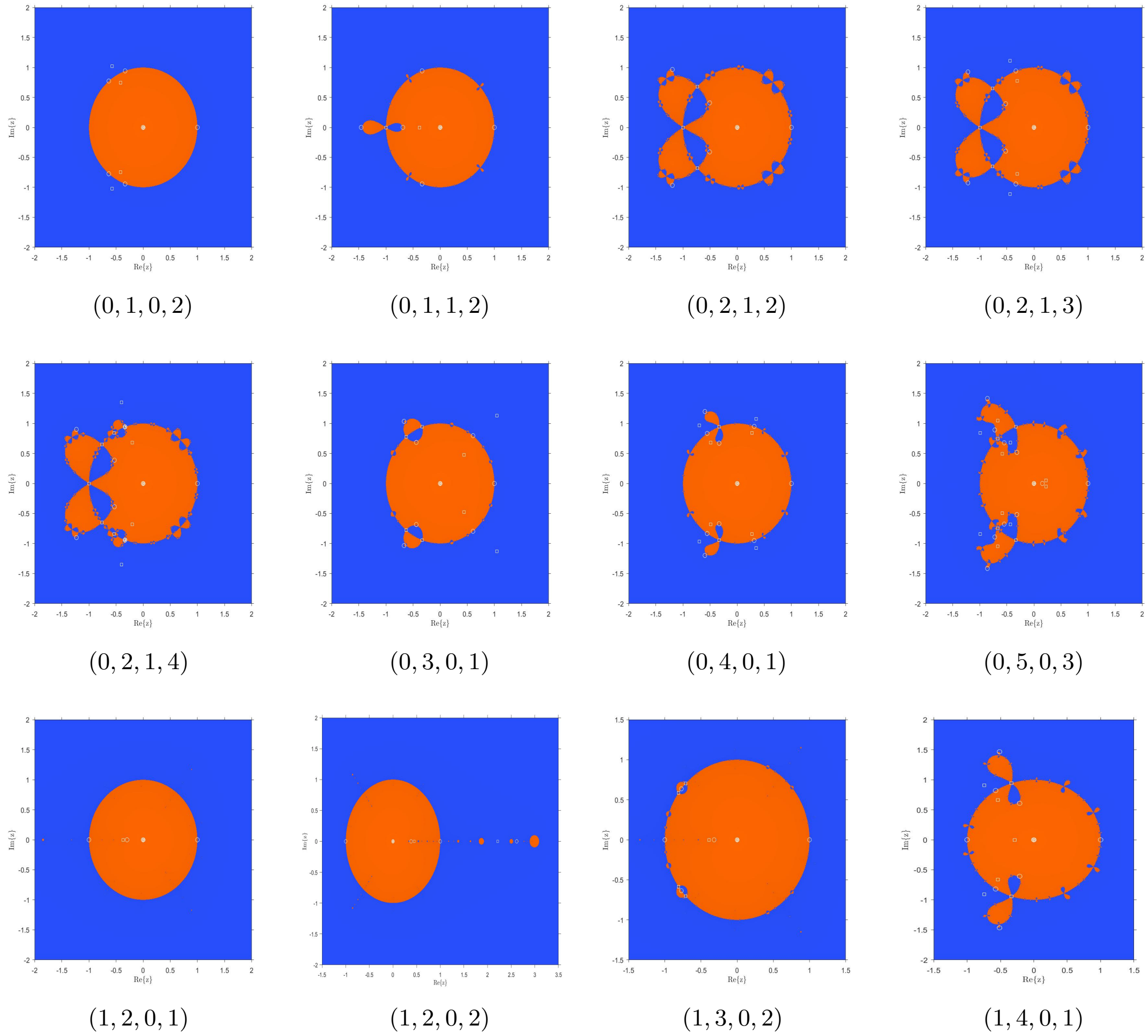
If in the iterative scheme (3) the weight function $H[\mu(x)]$ satisfies the conditions of Theorem 2.1, then for fixed values of k, g, l, and w, an optimal fourth-order iterative method is obtained in the sense of Kung and Traub. We are interested in determining its stability over certain simple test functions, such as quadratic polynomials. Indeed, given an initial estimate x_0, we seek to determine the final state of its orbit, i.e., to which attracting fixed point its orbit converges. It can be proven that scheme (3) satisfies the Scaling Theorem (see [14] for the Scaling Theorem of Newton's method). Therefore, through analytic conjugation, the rational operator of the iterative method applied to any polynomial $p(z) = (z - a)(z - b)$ has the same asymptotic behavior as the associated operator obtained by applying the conjugation $\phi(z) = \frac{z-a}{z-b}$, where $\phi(z)$ is a Möbius transformation satisfying the following properties: $\phi(\infty) = 1$, $\phi(a) = 0$, and $\phi(b) = \infty$. In this sense, we can perform a dynamical analysis on an associated operator that does not depend on the roots a and b. The codes used to present the dynamical planes are presented in Chicharro et al. [15].

The basin of attraction for the root $z = 0$ is represented in orange color, while the basin for $z = \infty$ is represented in blue color. Additionally, we assign different colors like green, red, etc., depending on the number of attracting fixed points associated with γ_0, and we use black color to represent basins of periodic orbits.

To construct the dynamical planes, we define a grid in the complex plane with a 2000×2000 point grid, where each point corresponds to a different value of the initial estimation z_0. Each dynamical plane shows the final state of the orbit of each point on the grid, considering a maximum of 200 iterations and a tolerance of 10^{-3}.

In Figure 1, we present the dynamic planes of the operators associated with some of the values of (k, g, l, w) that exhibit better stability on quadratic polynomials.

Figure 1: Dynamical planes of some stable (k, g, l, w) values.



These dynamical planes emphasize the favorable behavior in terms of stability of certain members of the family of iterative schemes (3) when applied to quadratic polynomials. This characteristic has evolved into a practical indicator for discerning between the different members of such a family, as those with good performance tend to outperform the unstable ones when confronted with more complex nonlinear equations.

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