Doubly quasi-stochastic and strictly checkerboard matrices of order 3

B. Cantó,[1] R. Cantó and A. M. Urbano

I.U. de Matemática Multidisciplinar, Universitat Politècnica de València
Camí de Vera s/n, València, Spain.

Abstract

In this work, doubly quasi-stochastic and strictly checkerboard matrices of order 3 are considered. In particular, these matrices are used as combined matrices, and their properties play an important role in this study. The objective is to prove the existence of nonsingular matrices of order 3, whose combined matrix exhibits quasi-stochastic and strictly checkerboard properties, and to construct such matrices. Through rigorous analysis and mathematical construction, we demonstrate the feasibility of this proposition.

1 Introduction

A real matrix $U = (u_{ij})$ of order n is a doubly quasi-stochastic matrix if the values in each row and each column sum up to 1, [1]. If the matrix is nonnegative matrix, it is called a doubly stochastic matrix. These matrices find applications in various fields, including the study of Markov chains [2], parameters estimation in doubly stochastic block models for network data analysis [3], and representation of certain quantum operations [4]. Some works on doubly quasi-stochastic matrices have been found in [5], where they appear as the natural generalization of doubly stochastic matrices and in [1], where a characterization of real matrices such that their combined matrix is quasi-stochastic of order 3 is obtained.

The combined matrix of a nonsingular matrix $A = (a_{ij})$ is defined as $U = \mathcal{C}(A) = A \circ A^{-T}$ where $\circ$ denotes the Hadamard (entrywise) product, and A^{-T} represents the inverse transpose, $(A^{-1})^T$, of A. Some properties of combined matrices have been studied in mathematics. For instance, the problem of characterizing when three real numbers are the diagonal entries of a combined matrix can be found in [6, 7]. Moreover, since the sum of the entries of each row and each column of a combined matrix is 1, it is a doubly quasi-stochastic matrix. In control theory, the combined matrix of a real matrix is known as the Relative Gain Array matrix (RGA), and it is used to determine the best pairing that corresponds to good controller performance. That is, the RGA is a normalized form of the gain matrix that describes the impact of each control variable on the output relative to each control variable's impact on other variables. The process interaction of open-loop and closed-loop control systems is measured for all possible input-output variable pairings. A ratio of this open-loop 'gain' to this closed-loop 'gain' is determined, and the results are displayed in the RGA matrix, see [8].

[1] bcanto@mat.upv.es

A real matrix $B = (b_{ij})$ of order n is a checkerboard matrix if $\text{sign}(b_{ij}) = (-1)^{i+j}$ or $b_{ij} = 0 \; \forall \, i, j = 1, \ldots, n$. B is a strictly checkerboard matrix when no entry is zero [9]. The signature of matrix B, denoted as $\sigma_B = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, shows valuable information about the sign of its determinants with σ_i, $i = 1, 2 \ldots, n$, which can each take on values of 1 or -1. Specifically, $\sigma_1 = 1$ implies that all entries of the matrix are nonnegative, while $\sigma_1 = -1$ indicates the converse. Similarly, σ_i represents the sign of all minors of order i, $i = 2, 3, \ldots, n-1$ and σ_n corresponds to the determinant of B.

This work focuses on doubly quasi-stochastic and strictly checkerboard matrices U of order 3. In this case, the matrices A such that $\mathcal{C}(A) = U$ have the following signatures: $\sigma_A = \{1, \ 1, \ 1\}$, $\sigma_A = \{-1, \ 1, \ -1\}$, $\sigma_A = \{1, \ -1, \ -1\}$ or $\sigma_A = \{-1, \ -1, \ 1\}$. Note that if $\sigma_A = \{1, \ 1, \ 1\}$ then $\sigma_{-A} = \{-1, \ 1, \ -1\}$ and since $\mathcal{C}(A) = \mathcal{C}(-A)$, the results are the same for both. Several authors have studied the matrices A such that $\sigma_A = \{1, \ 1, \ 1\}$, that is, totally positive matrices, for example, [7] or [1]. In [7], necessary and sufficient conditions for three real numbers greater than 1 to be the entries on the main diagonal of the combined matrix of a totally positive matrix A are given, and a complex method for obtaining A is proposed. In [1], some conditions that the entries of a specific doubly quasi-stochastic matrix of order 3 must satisfy in order to exist a real matrix such that its combined matrix is that one are given, and an easy technique to obtain a totally positive matrix from the study of the variation interval of one of its entries is provided.

Using a similar technique as in [1], we want to achieve the objective of this work. It is to determine if there exists a nonsingular matrix A whose signatures are $\sigma_A = \{1, \ -1, \ -1\}$ or $\sigma_A = \{-1, \ -1, \ 1\}$, such that its combined matrix is a doubly quasi-stochastic and checkerboard matrix and to construct it.

2 Results

In this section, we consider the combined matrix $U = \mathcal{C}(A)$. The following Lemma gives basic properties of combined matrices that we use in this work.

Lemma 1. *[10] The combined matrix $\mathcal{C}(A) = (c_{ij}) = A \circ A^{-T}$ of a nonsingular matrix $A = (a_{ij})$ satisfies*

(a) If D_1 and D_2 are two diagonal nonsingular matrices then $\mathcal{C}(A) = \mathcal{C}(D_1 A D_2)$.

(b) If P and Q are two permutation matrices, then $\mathcal{C}(PAQ) = P\mathcal{C}(A)Q$.

(c) $\mathcal{C}(A^{-T}) = \mathcal{C}(A)$.

(d) If $c_{ij} \neq 0$ then $a_{ij} \neq 0$, $\forall \ i, j$.

In this work, the matrix $U = (u_{ij})$ does not have zero entries. Then, we can use the following result from [1].

Theorem 1. *Let $U = (u_{ij})$ be a doubly quasi-stochastic matrix of order 3 without zero entries in its first row and its first column. Then, there exists a nonsingular matrix $A = (a_{ij})$, such that $\mathcal{C}(A) = U$, if and only if the polynomial*

$$P(x) = u_{11}x^2 + (u_{22} - u_{13}u_{31} - u_{33}u_{11})x + u_{33}u_{13}u_{31} \tag{1}$$

has at least a real root. Additionally, if $u_{21} + u_{31} \neq 0$, the real root is different from $\dfrac{-u_{13}u_{31}}{u_{21} + u_{31}}$.

If $u_{ii} > 1$, for $i = 1, 2, 3$, $u_{11} - u_{22} + u_{33} - 1 = -K < 0$ and $u_{13} \in]0, K[$, U is a doubly quasi-stochastic and checkerboard matrix

$$U = \begin{pmatrix} u_{11} & 1 - u_{11} - u_{13} & u_{13} \\ 1 - u_{11} + u_{13} - K & u_{22} & 1 - u_{13} - u_{33} \\ K - u_{13} & 1 - u_{33} + u_{13} - K & u_{33} \end{pmatrix}.$$

Moreover, if $u_{13} \notin \mathcal{J}$ where $\mathcal{J} = \{0, K, 1 - u_{11}, K + u_{11} - 1, 1 - u_{33}, K + u_{33} - 1\}$, then U is strictly checkerboard.

Since U is a doubly quasi-stochastic and strictly checkerboard matrix, by Lemma 1, if $A = (a_{ij})$ satisfies that $\mathcal{C}(A) = U$, then $a_{ij} \neq 0$, $\forall\, i, j$, and it can be factorized $A = D_1^{-1} T D_2^{-1}$ where D_1 and D_2 are nonsingular diagonal matrices and T and $S = T^{-1}$ are given by

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & t_{22} & t_{23} \\ 1 & t_{32} & t_{33} \end{pmatrix}, \quad S = \begin{pmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{13} - u_{21} + s_{33} & -u_{31} - s_{33} \\ u_{13} & -u_{13} - s_{33} & s_{33} \end{pmatrix}. \tag{2}$$

By Lemma 1, $\mathcal{C}(T) = \mathcal{C}(A) = U$, then, using a similar technique as in [1], we give the following result.

Theorem 2. *Necessary and sufficient conditions for a doubly quasi-stochastic and strictly checkerboard matrix $U = (u_{ij})$ of order 3 to be the combined matrix of a nonsingular matrix A with signature $\sigma_A = \{1, -1, -1\}$ are:*

1. *The entries u_{11}, u_{22} and u_{33} satisfy*

$$\begin{aligned} &u_{ii} > 1, \quad i = 1, 2, 3 \\ &u_{22} > max\{1, u_{11}, u_{33}, u_{11} + u_{33} - 1\}. \end{aligned} \tag{3}$$

2. *The entry $u_{13} \in]\,0,\ K\,[$, $K = -(u_{11} - u_{22} + u_{33} - 1) = u_{13} + u_{31}$, satisfies*

$$u_{13}(K - u_{13}) \geq (\sqrt{u_{22}} + \sqrt{u_{11}u_{33}})^2. \tag{4}$$

Proof. Let $U = (u_{ij})$ be a doubly quasi-stochastic and strictly checkerboard matrix of order 3 where its entries satisfy the conditions given in (3) and (4). Applying Theorem 3 of [1], there exists a matrix A such that $\mathcal{C}(A) = U$. We rewrite the polynomial (1) in terms of K as

$$P(x) = u_{11}x^2 + (u_{22} - u_{13}(K - u_{13}) - u_{11}u_{33})x + u_{33}u_{13}(K - u_{13})$$

and obtain its real roots. They are positive. Thus, the matrices S and T given in (2) are a checkerboard matrix and a positive matrices, respectively. To determine if $\sigma_T = \{1, -1, -1\}$, we study the sign of $\det(S)$, and to do so, we study the sign of $\det(S[1, 3])$.

Let x_1 be the real root of $P(x)$ obtained by adding a partial sum of the square root. Then

$$\begin{aligned} \det(S[1,3]) &= \frac{-u_{22} + u_{11}u_{33} - u_{13}(K - u_{13})}{2} \\ &\pm \frac{\sqrt{(u_{22} - u_{13}(K - u_{13}) - u_{11}u_{33})^2 - 4u_{11}u_{33}u_{13}(K - u_{13})}}{2}. \end{aligned}$$

Since

$$-u_{22} + u_{11}u_{33} - u_{13}(K - u_{13}) \quad < \quad -2(u_{22} + \sqrt{u_{11}u_{22}u_{33}}) < 0.$$

Using a technique similar to that used in [1], we obtain $\det(S[1,3]) < 0$. Thus, $\det(S) < 0$ and therefore $\sigma_T = \{1, -1, -1\}$.

From T and using diagonal matrices D_1 and D_2, we obtain a diagonal equivalent totally positive A such that $\mathcal{C}(A) = U$, with $\sigma_A = \{1, -1, -1\}$.

Conversely, let T be a sign regular matrix of order 3 with $\sigma_T = \{1, -1, -1\}$ given in (2) such that $0 < t_{ij} < 1$, $i, j = 2, 3$, $t_{33} < \min\{t_{23}, t_{32}\}$ and $t_{22} > \max\{t_{23}, t_{32}\}$. Thus, $0 < t_{33} < t_{22} < 1$. Let $U = (u_{ij})$ be the combined matrix of T. Considering that $\sigma_T = \{1, -1, -1\}$, then U is a doubly quasi-stochastic and checkerboard matrix. Applying [1], we have $u_{11} + u_{33} - 1 < u_{22}$.

Since $\det(T) = t_{22}(t_{33} - 1) + t_{23}(1 - t_{32}) + (t_{32} - t_{33}) < 0$ then $t_{22}(t_{33} - 1) < \det(T) < 0$ and $u_{22} = \dfrac{t_{22}(t_{33} - 1)}{\det(T)} > 1$. Moreover

$$\begin{aligned} u_{11} &= \frac{t_{22}t_{33} - t_{23}t_{32}}{\det(T)} < \frac{t_{22}(t_{33} - 1)}{\det(T)} = u_{22} \\ u_{33} &= \frac{t_{33}(t_{22} - 1)}{\det(T)} < \frac{t_{22}(t_{33} - 1)}{\det(T)} = u_{22}. \end{aligned}$$

Thus, $u_{22} > \max\{1, u_{11}, u_{22}, u_{11} + u_{33} - 1\}$.

Now, we prove that $u_{13}(K - u_{13}) \geq (\sqrt{u_{22}} + \sqrt{u_{11}u_{33}})^2$ using reduction ad absurdum. We suppose that

$$u_{13}(K - u_{13}) < (\sqrt{u_{22}} + \sqrt{u_{11}u_{33}})^2 = u_{22} + 2\sqrt{u_{11}u_{22}u_{33}} + u_{11}u_{33}$$

holds. Then,

$$2\sqrt{u_{11}u_{22}u_{33}} > u_{13}(K - u_{13}) - u_{11}u_{33} - u_{22}.$$

If $|u_{13}(K - u_{13}) - u_{11}u_{33} - u_{22}| < 2\sqrt{u_{11}u_{22}u_{33}}$ working in the same way that in [1], we obtain that $P(x)$ given in (1) has no real roots. As a consequence, does not exist any nonsingular matrix A such that $\mathcal{C}(A) = U$.

If $|u_{13}(K - u_{13}) - u_{11}u_{33} - u_{22}| > 2\sqrt{u_{11}u_{22}u_{33}}$ then

$$-u_{13}(K - u_{13}) + u_{11}u_{33} + u_{22} > 2\sqrt{u_{11}u_{22}u_{33}},$$

which implies that

$$u_{13}(K - u_{13}) < u_{11}u_{33} + u_{22} - 2\sqrt{u_{11}u_{22}u_{33}} = (\sqrt{u_{22}} - \sqrt{u_{11}u_{33}})^2.$$

In this case, and analogously to [1], we obtain $\det(S) > 0$. As a consequence, $\sigma_T \neq \{1, -1, -1\}$.

Thus, $u_{13}(K - u_{13}) \geq (\sqrt{u_{22}} + \sqrt{u_{11}u_{33}})^2$. □

Remark 1. *Since $-A$ has $\sigma_{-A} = \{-1, -1, 1\}$, the results obtained are the same. The combined matrix $U = \mathcal{C}(A) = \mathcal{C}(-A)$ is a doubly quasi-stochastic and checkerboard matrix of order 3.*

Example 1. *Consider the doubly quasi-stochastic and checkerboard matrix*

$$U = \begin{pmatrix} 1 & -u_{13} & u_{13} \\ -9 + u_{13} & 10 & -u_{13} \\ 9 - u_{13} & -9 + u_{13} & 1 \end{pmatrix}, \ \forall u_{13} \in]0,\, 9[.$$

Note that,

$$\begin{aligned} u_{11} + u_{33} - 1 &= 1 \\ u_{22} = 10 &> \max\{1, u_{11}, u_{11} + u_{33} - 1\} = 1 \\ (\sqrt{u_{22}} + \sqrt{u_{11}u_{22}})^2 &= 17.3246. \end{aligned}$$

Choosing $u_{13} = 6$, *we have* $u_{13}(9 - u_{13}) = 18 > 17.3246$ *and*

$$U = \begin{pmatrix} 1 & -6 & 6 \\ -3 & 10 & -6 \\ 3 & -3 & 1 \end{pmatrix}.$$

The polynomial $P(x)$ *given in (1) is* $P(x) = x^2 - 9x + 18$ *and it has the positive roots* $x_1 = 6$ *and* $x_2 = 3$. *As* $u_{21} + u_{31} = 0$ *we can construct the matrix* S *using* x_1 *or* x_2.

- *For* $x_1 = 6$, *we have the matrices*

$$S_1 = \begin{pmatrix} 1 & -3 & 3 \\ -6 & 15 & -9 \\ 6 & 12 & 6 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2/3 & 1/2 \\ 1 & 1/3 & 1/6 \end{pmatrix},$$

with $\det(S_1) = -18$, $\det(T_1) = -1/18$ *and* $\sigma_{T_1} = \{1, -1, -1\}$.

- *For* $x_2 = 3$, *we have the matrices*

$$S_2 = \begin{pmatrix} 1 & -3 & 3 \\ -6 & 12 & -6 \\ 6 & -9 & 3 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5/6 & 2/3 \\ 1 & 1/2 & 1/3 \end{pmatrix},$$

with $\det(S_2) = -18$, $\det(T_2) = -1/18$ *and* $\sigma_{T_1} = \{1, -1, -1\}$.

Note that,

$$\begin{aligned} \tilde{T_1} &= -T_1 = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -2/3 & -1/2 \\ -1 & -1/3 & -1/6 \end{pmatrix}, \ \det(\tilde{T_1}) = 18 \\ \tilde{T_2} &= -T_2 = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -5/6 & -2/3 \\ -1 & -1/2 & -1/3 \end{pmatrix}, \ \det(\tilde{T_2}) = 18, \end{aligned}$$

and, in these cases, $\sigma_{\tilde{T_1}} = \sigma_{\tilde{T_2}}\{-1, -1, 1\}$.

From T_1 *and* T_2 *we obtain the matrices* A *whose signature is* $\sigma_A = \{1, -1, -1\}$ *and from* $\tilde{T_1}$ *and* $\tilde{T_1}$ *we obtain the matrices* A *whose signature is* $\sigma_A = \{-1, -1, 1\}$, *using diagonal matrices* D_1 *and* D_2 *with all diagonal entries positive or negative.*

3 Conclusions

We have provided an answer to the question regarding the existence of nonsingular matrices A of order 3 with specific signatures, such that their combined matrices are both doubly quasi-stochastic and checkerboard properties. These matrices are characterized by signatures $\sigma_A = \{1, 1, 1\}$, $\sigma_A = \{-1, 1, -1\}$, $\sigma_A = \{1, -1, -1\}$ and $\sigma_A = \{-1, -1, 1\}$.

Through this investigation, the concepts of doubly quasi-stochastic and checkerboard matrices, as well as combined matrices of nonsingular matrices, have been defined. Moreover, as other researchers have previously studied the first two signatures, our focus has been on the signatures

$\sigma_A = \{1, -1, -1\}$ and $\sigma_A = \{-1, -1, 1\}$. Employing a similar technique to those used by prior authors, we have given a result on the existence of nonsingular matrices with these signatures, yielding combined matrices that possess both desired properties, that is doubly quasi-stochastic and checkerboard matrices.

In conclusion, our research shows the relationships between certain types of signatures of nonsingular matrices and the properties of their combined matrices, enriching the discourse on matrix theory.

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