About the stability of self-accelerating parameters in vectorial iterative methods without memory

Alicia Cordero[a],[1] Antmel Rodríguez cabral[b], Javier G. Maímo[b] and Juan R. Torregrosa[a]

(a) I.U. de Matemática Multidisciplinar, Universitat Politècnica de València
Camí de Vera s/n, València, Spain.

(b) Area de Ciencias Básicas y Ambientales (CBA), Instituto Tecnológico de Santo Domingo (INTEC), Santo Domingo, Dominican Republic.

1 Introduction

This study investigates the stability role of self-accelerating parameters in vectorial iterative methods without memory, building on the prior work by Singh, Sharma, and Kumar [1], who introduced an innovative biparametric scheme. Our research focuses on evaluating two subfamilies derived from the original model: the first with $p_1 = 1$ and arbitrary p_2, and the second with arbitrary p_1 and $p_2 = 1$, to compare their behavior and effectiveness in solving nonlinear systems.

A detailed stability analysis is performed through dynamic and bifurcation planes. This analysis involves observing how the parameters p_1 and p_2 influence the method's stability and convergence speed, using dynamical graphs to illustrate similarities and differences between those subfamilies. These graphs help identify sets of parameter values where the methods are more stable and the conditions under which chaotic behaviors or strange fixed points can emerge.

Moreover, the mathematical properties of fixed points and periodic orbits within this context are described, including their classification and stability. Attractors and repellers are studied, and how parameter selection affects the presence and type of these fixed points, offering a deeper insight into the real multidimensional dynamics of these systems.

Numerical testing is another crucial component of our study. It is implemented on various nonlinear systems, showing that modifications to the iterative schemes not only preserve computational efficiency but also enhance convergence. These results are verified against the expected theoretical values, demonstrating the validity and robustness of the proposed schemes under different parameter settings.

[1]acordero@mat.upv.es

2 Methods

Singh and Sharma et. al in [1] recently presented a interesting biparametric family with a novel approach by introducing the factor $\frac{F(w^{(k)})^T F(w^{(k)})}{F(x^{(k)})^T F(x^{(k)})}$ that can be viewed as $F(w^{(k)})^T F(w^{(k)}) = \left\|F(w^{(k)})\right\|^2$, and $F(x^{(k)})^T F(x^{(k)}) = \left\|F(x^{(k)})\right\|^2$, respectively.

The proposed scheme had the following iterative expression:

$$w^{(k)} = x^{(k)} - F'\left(x^{(k)}\right)^{-1} F\left(x^{(k)}\right), \qquad k = 0,1,2,\ldots \tag{1}$$

$$x^{(k+1)} = w^{(k)} - \left(p_1 + p_2 \frac{F\left(w^{(k)}\right)^T F\left(w^{(k)}\right)}{F\left(x^{(k)}\right)^T F\left(x^{(k)}\right)}\right) F'\left(w^{(k)}\right)^{-1} F\left(w^{(k)}\right) \tag{2}$$

where the parameters p_1 and p_2 are real numbers. The method consists of using the result of multiplying a row vector function by a column vector function, which results in a scalar function. This approach allows a significant reduction in the work effort, leading to a considerable reduction in the costs associated with computational processing.

Specifically, we work with the following schemes:

$$\begin{aligned} w^{(k)} &= x^{(k)} - F'\left(x^{(k)}\right)^{-1} F\left(x^{(k)}\right), \\ x^{(k+1)} &= w^{(k)} - \left(1 + p_2 \frac{F\left(w^{(k)}\right)^T F\left(w^{(k)}\right)}{F\left(x^{(k)}\right)^T F\left(x^{(k)}\right)}\right) F'\left(w^{(k)}\right)^{-1} F\left(w^{(k)}\right). \end{aligned} \tag{3}$$

and

$$\begin{aligned} w^{(k)} &= x^{(k)} - F'\left(x^{(k)}\right)^{-1} F\left(x^{(k)}\right), \\ x^{(k+1)} &= w^{(k)} - \left(p_1 + \frac{F\left(w^{(k)}\right)^T F\left(w^{(k)}\right)}{F\left(x^{(k)}\right)^T F\left(x^{(k)}\right)}\right) F'\left(w^{(k)}\right)^{-1} F\left(w^{(k)}\right). \end{aligned} \tag{4}$$

Where 3 is the scheme obtained with $p_1 = 1$ and arbitrary p_2 which will have a convergence order of 4 for the members of this subfamily except for $p_2 = 1$ which would be 5, while 4 is the subfamily obtained by arbitrary p_1 and $p_2 = 1$ whose convergence order for the members is 2 except for $p_1 = 1$ which is 5.

3 Results

The numerical results obtained were performed with the Matlab2022b version with 2000-unit precision arithmetics. Now, we describe some important data that appear in the tables:

- k= Number of iterations performed ("-" appears if there is no convergence or it exceeds the maximum number of iterations allowed).
- $\bar{x}$= which solution was reached by the iterative method with the given initial estimation.
- ρ (approximate computational order convergence) defined by Cordero and Torregrosa in [5]

$$\rho = \frac{\ln \frac{\left\|x^{(k+1)} - x^{(k)}\right\|}{\left\|x^{(k)} - x^{(k-1)}\right\|}}{\ln \frac{\left\|x^{(k)} - x^{(k-1)}\right\|}{\left\|x^{(k-1)} - x^{(k-2)}\right\|}}, \qquad \text{for each } k = 2,3,\ldots,$$

(if the value of ρ for the last iterations is not stable, then "-" appears in the table).

- ϵ_{aprox}= The error obtained by taking the norm of the solution in one iteration and its prior, $\left\|x^{(k+1)} - x^{(k)}\right\|$,for the table, only the one obtained in the last two iterations will be placed.
- ϵ_f= Norm of the function evaluated in the following estimation, $\left\|F\left(x^{k+1}\right)\right\|$ (if the error estimates are very far from zero or we get NAN, infinity, then we will place "-")

For the stopping criterion, the first of the three items to be fulfilled first was taken into account

Example 1 *The first test is the following system,*

$$\left\{\begin{array}{rcl} e^{x_1}e^{x_2} + x_1 cos(x_2) & = & 0, \\ x_1 + x_2 - 1 & = & 0. \end{array}\right. \tag{5}$$

The exact solution of system (5) *is:*

$\bar{x}_1 \approx (-185.93938,\ 186.93938)$; $\bar{x}_2 \approx (-50.78272,\ 51.78272)$; $\bar{x}_3 \approx (24.45053,\ -23.45053)$
$\bar{x}_4 \approx (143.92357\ \ -142.92357)$; $\bar{x}_5 \approx (165.94999, 164.94999)$,
and we use as initial estimation $x^{(0)} = (-2,\ 3)$.

Example 2 *The second example is a system of equations of size* $n = 20$*, whose expression is:*

$$\arctan(x_i) + 1 - 2\left[\left(\sum_{k=1}^{20} x_k^2\right) - x_i^2\right] = 0, \quad i = 1, 2, \ldots 20. \tag{6}$$

The solution of (6) *is* $\bar{x}_1 \approx (0.175768, \ldots, 0.175768)^T$, $\bar{x}_2 \approx (-0.1496, \ldots, -0.1496)^T$ *and we use as initial estimation* $x^{(0)} = (1, 1, \ldots, 1)^T$.

Table 1: Numerical results for Example 1.

Iterative methods class (1)	k	ρ	ϵ_{aprox}	ϵ_f	$\bar{x}$
$p_2 = -7$	6	4	5.473×10^{-174}	9.164×10^{-696}	$\bar{x}_2$
$p_2 = 1$	6	5	4.921×10^{-470}	1.834×10^{-2006}	$\bar{x}_3$
$p_2 = -52$	16	4	2.477×10^{-300}	6.687×10^{-1204}	$\bar{x}_1$
$p_2 = 15.983$	6	4	6.334×10^{-263}	1.015×10^{-1054}	$\bar{x}_5$
$p_2 = 15.20$	7	4	8.199×10^{-216}	1.017×10^{-863}	$\bar{x}_4$

Table 2: Numerical results for Example 2.

Iterative methods class (2)	k	ρ	ϵ_{aprox}	ϵ_f	$\bar{x}$
$p_1 = -1$	14	2	7.906×10^{-159}	1.067×10^{-315}	$\bar{x}_1$
$p_1 = 1$	6	5	4.326×10^{-319}	1.69×10^{-1591}	$\bar{x}_1$
$p_1 = 2$	15	2	2.154×10^{-161}	3.926×10^{-321}	$\bar{x}_2$
$p_1 = -25$	-	-	-	-	-
$p_1 = 8$	-	-	-	-	-

On the other hand, the bifurcation planes show how the behavior of a dynamic system changes as the parameters are varied. This allows to identify regions of stability and chaos, or transitions between different types of behavior, such as convergence to a solution or divergence while dynamical diagrams help to analyze the stability of solutions found by iterative methods. For example, a diagram can show whether iterations stabilize at a fixed point, converge to a limit cycle, or exhibit chaotic behavior, depending on the initial conditions and parameters of the method.

Some bifurcation and dynamical planes are shown below to demonstrate the behavior of some members of the subfamilies.

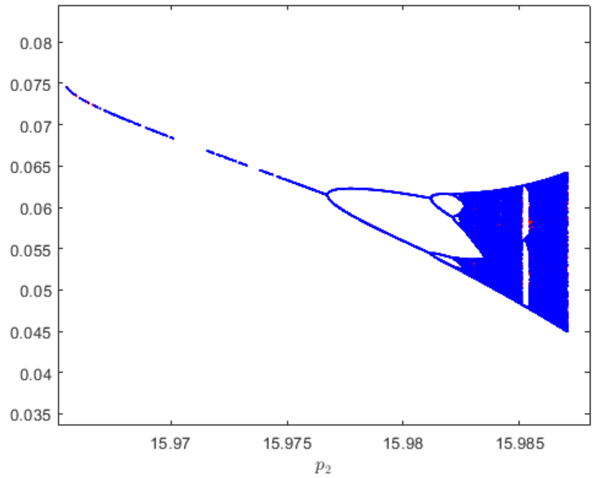

Figure 1: Feigenbaum diagrams of $SS(x, p_2) \quad for\, p_2 > 0$.

In the dynamic plane presented in Figure 3, we observe a complex pattern of colors representing basins of attraction towards different fixed points in the context of an iterative system. Each distinct color, except for black, indicates a region where initial conditions converge towards a specific fixed point. This representation is a key visual tool for understanding how different initial conditions affect the behavior of the system under study.

The color black on the plane represents areas where initial conditions lead to divergence or extremely slow convergence towards a fixed point. This implies that, from these initial conditions, the iterative method does not stabilize quickly or may diverge.

On the other hand, the yellow lines and dots represent the orbit described by a specific initial estimate. This color indicates the presence of a periodic orbit, where the initial condition does not converge to a fixed point but enters a repetitive cycle. This type of behavior is crucial for dynamic analysis, as periodic orbits can indicate more complex behaviors, like chaos, under slightly different parameter settings.

The interpretation of these colors and shapes on the plane allows researchers and analysts to determine the effectiveness of different iterative methods and adjust parameters to achieve the desired convergence. Additionally, it facilitates the identification of problematic regions that require special attention, either in terms of numerical stability or in the search for more precise solutions.

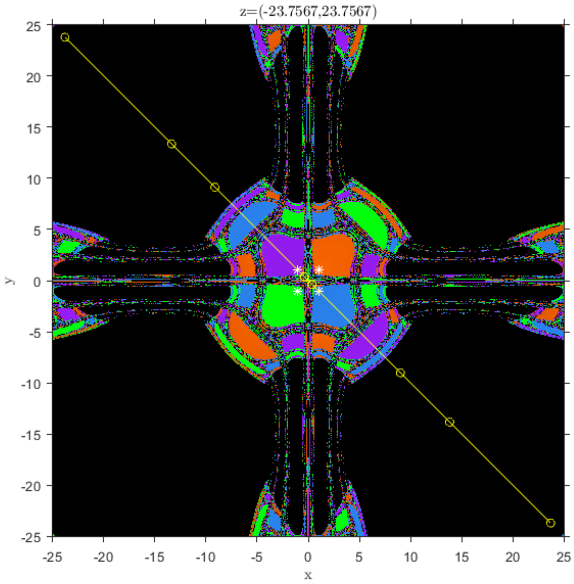

Figure 2: Periodic orbits for p_2 represented in dynamical planes

4 Conclusions

This comprehensive analysis proves that the proper choice of parameters is crucial for optimizing the stability and efficacy of memoryless iterative methods. Furthermore, it provides a solid foundation for future research that may explore more complex configurations or different types of nonlinear systems of equations.

In conclusion, this work not only highlights the importance of careful parameter selection in iterative methods but also establishes a clear methodology for assessing the stability and dynamics of these schemes, making a significant contribution to the existing literature and offering new perspectives for the efficient resolution of nonlinear problems in mathematical practice.

References

[1] H. Singh, J. R. Sharma, S. S. Kumar, A simple yet efficient two-step fifth-order weighted-newton method for nonlinear models, *Numerical Algorithms*, Vol.93, 203–225, (2022) .

[2] R. C. Robinson, An introduction to dynamical systems: continuous and discrete, Vol. 19, *American Mathematical Soc.*, (2012).

[3] G. Julia, Mémoire sur l'itération des fonctions rationnelles, *Journal de Mathématiques Pures et Appliquées*, Vol.1,47–245, (1918).

[4] P. Fatou, Sur les équations fonctionnelles, Bulletin de la Société Mathématique de France Vol.47, 161–271, (1919) .

[5] A. Cordero, J. R. Torregrosa, Variants of Newton's method using fifth-order quadrature formulas, *Applied Mathematics and Computation*, Vol. 190 (1), 686–698, (2007) .