# Dynamical study of a family for solving nonlinear equations with multiple roots

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## 1.1 Introduction

Iterative methods for solving nonlinear equations $f(x) = 0$, where $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, are a useful technique when the analytical methods are not able to give a closed-form solution.

The classical way to compare iterative methods is focused on their order of convergence, the number of functional evaluations per step and the optimality [1], amongst others. However, in recent decades a trend of stability analysis of iterative methods has been developed, understood as the set of initial estimates that manage to converge to solutions. The first appearance of convergence basins in the complex plane dates back to 1879 [2], where Cayley studies the convergence basins of Newton's method applied on the polynomial $p(z) = (z - a)(z - b)$. With the revolution of computation in the middle of the 20th century, numerical analysis acquired great relevance due to its implementability, and stability analysis was resumed with the help of informatics.

Numerous stability investigations of methods for obtaining single roots of equations or single roots of systems of equations can be found in the literature [3]. However, stability studies for methods that obtain multiple roots are not so frequent [4].

In this work, a dynamical study for an iterative method devoted to find multiple zeros of a nonlinear equation is performed. The iterative procedure under analysis is a derivative-free two-step method based on weight functions, whose expression is

$$\begin{aligned} y_k &= x_k - m\frac{f(x_k)}{f[w_k,x_k]}, \\ x_{k+1} &= x_k - mH(t_k)\frac{f(x_k)}{f[w_k,x_k]}, \quad k = 0, 1, \ldots, \end{aligned} \tag{1.1}$$

where $w = x + (f(x))^2$, $m$ is the multiplicity of the root, $t = \sqrt[m]{\frac{f(y)}{f(x)}}$ and $H(t)$ is a weight function that satisfies $H(0) = H'(0) = 1$ and $H''(0) = 4$. The order of convergence of the family under these weight function conditions is four. For the dynamical study, we are chosing the polynomial function $H_1(t) = 1 + t + 2t^2 + \gamma t^3$. The following section discusses this study to determine the most stable members of the family when applied to solving quadratic polynomials with a root of multiplicity $m = 2$.

## 1.2 Dynamical analysis

The foundations of dynamical complex analysis [5] are applied to analyse the stability of the iterative method. Since the method is derivative-free, applying a scaling theorem is not possible [6]. Therefore, we are applying method (1.1) on the multiple-root function $p(z) = (z - 1)^2$.

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The resulting operator using the weight function $H_1(t)$ can be simplified as

$$R(z) = \frac{N_{13}(\gamma, z)}{\left(z^3 - 3\,z^2 + 3\,z + 1\right)^4}, \tag{1.2}$$

where $N_{13}(\gamma, z)$ is a polynomial of degree 13.

The fixed points of $R(z)$ are the double root of $p(z)$, $z^* = 1$, being superattracting, and the strange fixed points $z^F_{1-9}$, which are the roots of the ninth-degree polynomial:

$$q_\gamma(z) = \gamma z^9 + 4z^9 - 9\gamma z^8 - 36z^8 + 36\gamma z^7 + 144z^7 - 84\gamma z^6 - 322.$$

When the rational operator depends on a complex parameter, the stability plane is used to represent the region where the strange fixed points are attracting, repelling or neutral. Given a strange fixed point that depends on the parameter, the real and imaginary parts of the parameter are represented on each axis on the stability plane. The points in the plane represent the values of the parameter that give rise to a strange fixed point with repelling (represented in white), or attracting or neutral (represented in black) behavior. Each strange fixed point has an associated stability plane, but the asymptotic behavior of all of them can be plotted on the same graph using the unified stability plane.

Figure 1.1 represents the unified stability plane of $z^F_{1-9}$. Let us remark that when a point is represented in black in the unified stability plane it is because for that value of $\gamma$ some strange fixed point has an attracting behavior.

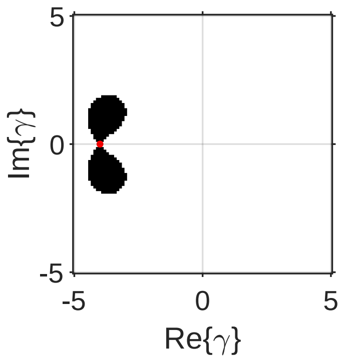


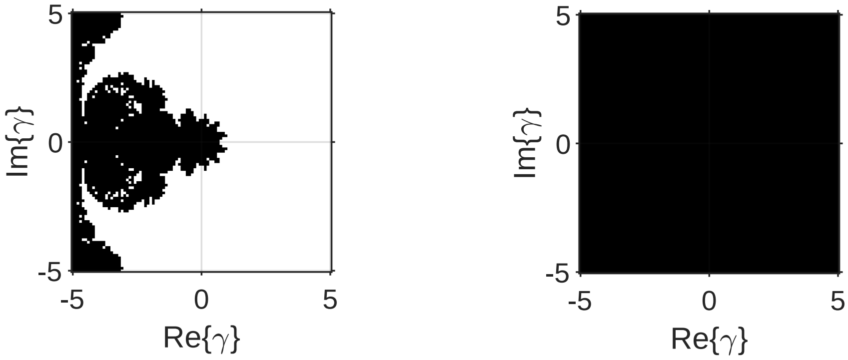

Figure 1.1: Unified stability plane. In red, $\gamma = -4$.

It can be observed in Figure 1.1 that most of the parameter values give rise to strange fixed points whose behavior is never an attractor, so that the only attractor would be the root of the polynomial. Also, it can be seen a region represented in black on the left side of the stability plane, whose parameter values should be avoided in order to obtain the most stable members of the iterative family.

The critical points of the operator are obtained by solving $R'(z) = 0$. In addition to the double root of $p(z)$, nine free critical points $z^C_{1-9}$ are obtained. The free critical points may belong to the basins of attraction of attracting periodic orbits instead of converging to the root of $p(z)$. Therefore, the location of the critical points of the system is also essential. To study the orbit of the critical points when the operator depends on at least one parameter, the parameter plane is used. The parameter plane associated to each free critical point allows to study the behavior of the class when the initial estimate is a critical point.

Figure 1.2 shows the unified parameters plane associated to the free critical points $z^C_{1-6}$ (Figure 1.2(a)) and for $z^C_{7-9}$ (Figure 1.2(b)).

It can be seen that the behavior is not as desired, since the parameter planes of $z^C_{7-9}$ there are no values of $\gamma$ whose orbit converges to the root. Figure 1.2(a) also shows parameter values that converge to other attracting orbits, but there is a large set of values that belong to the basins of attraction of $z^*$ taking as initial estimation those free critical points. We will rely on Figure 1.2(a) to select different methods from the family (corresponding to each value of $\gamma$) that can lead




(a) $z^C_{1-6}$



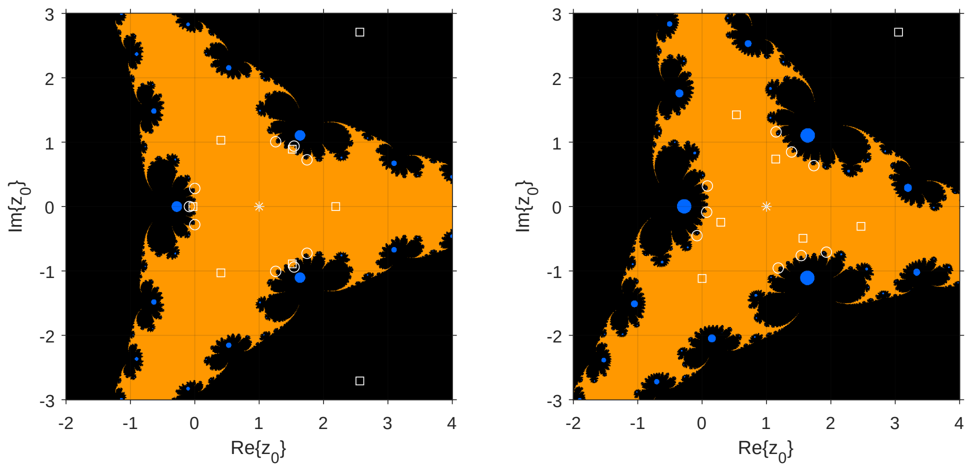
(b) $z^C_{7-9}$

Figure 1.2: Unified parameters planes of the free critical points.

to stable behavior. We will examine the convergence depending on the initial estimates using the dynamical planes of the selected methods.

The dynamical plane is a graphical tool for visualizing the basins of attraction of the operator's attracting points. Given a set of initial estimates, whose real and imaginary part is represented on the axis, each point in the plane is considered as initial guess to successively evaluate the rational operator and examine its convergence. The process ends when it converges to an attracting point or when a maximum number of allowed iterations is exceeded. In this work, the dynamical planes of the methods have been generated following the guidelines of [7]. Circles and squares represent the strange fixed points and the free critical points, respectively. Orange and blue colors represent the initial estimations that converge to $z^*$ and $z^F_\infty$, respectively. A tolerance of $10^{-3}$ and a maximum of 100 iterations has been established.

Figures 1.3 and 1.4 show different dynamical planes of the family of iterative schemes (1.1) when it is applied to $p(z)$. Let us remind that each value of parameter $\gamma$ provides an iterative scheme belonging to this family with different stability properties. These features can be visualized in the dynamical plane of each method.



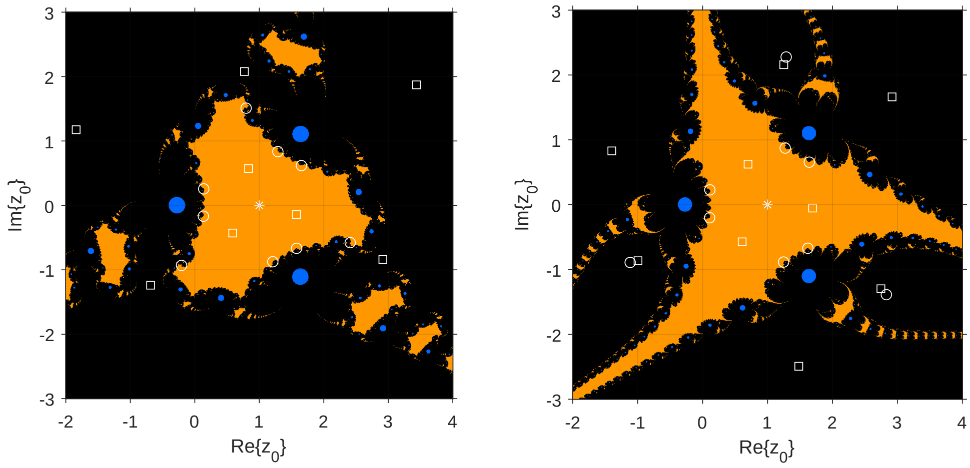

(a) $\gamma = 1$



(b) $\gamma = 2 - 3i$

Figure 1.3: Dynamical planes for $\gamma = 1$ and $\gamma = 2 - 3i$.

Figure 1.3 represents the dynamical planes associated to values of the parameter located in stability regions, that is, values of $\gamma$ located in the white regions of the unified stability plane and the unified parameters plane. These values correspond to iterative schemes for which all the strange fixed points are always repelling (Figure 1.1) and the free critical points $x^C_{1-6}$ belong to the basin




(a) $\gamma = -4.5 - 4.5i$ (b) $\gamma = -3.5 - i$

Figure 1.4: Dynamical planes for $\gamma = -4.5 - 4.5i$ and $\gamma = -3.5 - i$.

of attraction of $z^*$ (Figure 1.2(a)).

On the other hand, Figure 1.4 depicts the dynamical planes associated to values of the parameter of the less favorable regions of the previous dynamical study. Specifically, $\gamma = -4.5 - 4.5i$ is represented in white in Figure 1.1, while in Figure 1.2(a) it corresponds to a value of the parameter for which all the free critical points do not converge to the root of the polynomial (in black). In contrast, $\gamma = -3.5 - i$ corresponds to black regions in Figures 1.1 and 1.2. As a consequence, in both dynamical planes at Figure 1.4 the basins of attraction of the root are smaller than in the dynamical planes depicted in Figure 1.3. In both cases small basins of attraction of the infinity are observed.

Finally, we consider a particular case within the dynamical analysis of the family of iterative methods (1.1) when it is applied over polynomial $p(z)$. In particular, when $\gamma = -4$ the results are simplified and a stable method of the iterative class is obtained. This value of the parameter is represented in red on the unified stability plane of Figure 1.1.

For $\gamma = -4$, the fixed points of the resulting rational operator (1.2) and their asymptotic behaviour are

- $z^* = 1$, that is superattracting,
- $z_\infty^F = \infty$, that is parabollic, and
- $z_{1-6}^F$, that are repelling.

Let us remark that $z_{1-6}^F$ are three pairs of conjugate complex numbers. They are the roots of $7z^6 - 42z^5 + 105z^4 - 132z^3 + 81z^2 - 18z + 3 = 0$ and their approximate value is

$$z_{1,2}^F = 0.11471 \pm 0.214683i, \qquad z_{3,4}^F = 1.25672 \pm 0.874025i, \qquad z_{5,6}^F = 1.62857 \pm 0.659342i.$$

Although all the strange fixed points are always repelling, there are 9 free critical points that perturbate the stability of the method. These points correspond to the solutions of $z^9 - 9z^8 + 36z^7 - 74z^6 + 66z^5 + 24z^4 + 64z^3 - 426z^2 + 489z - 227 = 0$. Figure 1.5 represents the dynamical plane of this single method.

The dynamical plane shows the convergence to the multiple root for a wide set of initial estimates. Some free critical points belong to the basin of attraction of $z^*$. Yellow lines show the orbit of specific free critical points, showing their trend to reach $z_\infty^F$ as iterations increase.

## 1.3 Conclusions

In this work, an analysis of the stability of an iterative family with a parameter weight function has been carried out. Complex dynamical tools have been used to carry out this study. The behaviour

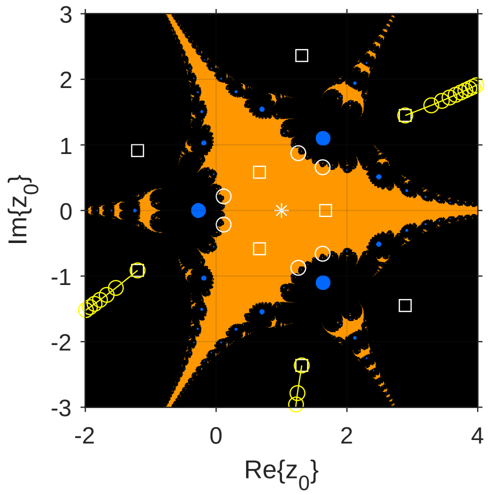



Figure 1.5: Dynamical plane of method for $\gamma = -4$.

of the iterative family when applied to a quadratic polynomial with a root of double multiplicity has been discussed. This analysis makes it possible to select the family members, i.e. the values of the parameter of the weight function, which give rise to the most stable iterative schemes. The basins of attraction of the root of the quadratic polynomial have also been visualised for some of the methods on their dynamical planes. This graphical tool has made it possible to visualise all the initial estimates of the methods which, after the iterative process, converge to the solution. Finally, the basins of attraction obtained for the iterative method corresponding to $\gamma = -4$ for which all the strange fixed points are repelling have been shown, and the convergence of the free critical points on their dynamical plane has been also studied.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