Random non-local Fisher-KPP population models

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1 Introduction

The use of non-local Fisher-KPP models in order to determine the population density $u(x,t)$ in the population dynamics take into account the consumption of resources and competitions, that they are rarely fully local, and modelling the depletion of resources by the spatial average represented by a convolution. But because coefficients and initial conditions in these models may be subject to uncertainties not only due to difficulties to measurement but also due to errors we introduce uncertainty, in the mean square (m.s.) sense, making these models more realistic. The random non-local Fisher-KPP model studied is important in the field of ecology and conservation biology. Assuming uncertainty motivates the consideration of the following random partial integro-differential reaction-diffusion problem (RPIDE)

$$u_t(x,t;\omega) = d(\omega)\, u_{xx}(x,t;\omega) + \beta(\omega)\, u(x,t;\omega)\left[1 - a(\omega)\, u(x,t;\omega) - b(\omega)\ (\psi * u)\,(x,t;\omega)\right], \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ . \quad (1)$$

The unknown $u(x,t;\omega)$, for an event $\omega \in \Omega$ on the probabilistic space $(\Omega, \mathcal{F}, \mathcal{P})$, represents the random population density of the considered species depending on both variables space x and time t. In (1) the convolution stochastic process (s.p.), $(\psi * u)\,(x,t;\omega)$, denotes

$$(\psi * u)\,(x,t;\omega) = \int_{\mathbb{R}} \psi(x-y)\, u(y,t;\omega)\, \mathrm{d}y\,, \qquad \text{for } t \text{ fixed}\,, \quad \omega \in \Omega\,, \quad (2)$$

where the kernel $\psi(x)$ is considered to be a non-negative continuous function compactly supported in the region $[-\delta, \delta]$, $\delta \in \mathbb{R}$, such that $\psi(-\delta) = \psi(\delta) = 0$, satisfying $\int_{-\delta}^{\delta} \psi(x)\ \mathrm{d}\,x = 1$, see [2]. It can be demonstrated that the convolution s.p. of the kernel $\psi(x)$ and the s.p. $u(x,t;\omega)$, for a fixed $t \in \mathbb{R}^+$ given by (2) is well-defined by m.s. convergent integral under the following conditions:

- $u(x,t;\omega)$ is a s.p. m.s. locally integrable and m.s. absolutely integrable, for a fixed $t \in \mathbb{R}^+$,
- and verifying

$$\int_{-\infty}^{\infty} \left(\|u(y,t;\omega)\|_2\right)^2\ \mathrm{d}y < +\infty\,, \quad \omega \in \Omega\,, \quad \text{for } t \text{ fixed}\,.$$

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We consider that the distribution of initial population is continuous positive deterministic function of compact support, given by $u(x,0)=f(x)$, $x\in[-\alpha,\alpha]$, $\alpha\in\mathbb{R}$, such that $f(-\alpha)=f(\alpha)=0$.

In RPIDE (1) the random input parameters $d(\omega)$, $\beta(\omega)$, $a(\omega)$ and $b(\omega)$, $\omega\in\Omega$, are strictly positive r.v.'s. having a *known uncertainty*, that is, they can follow a wide range of probability distributions such as beta, exponential, Gaussian, etc. For the sake of practical applications we assume that the uncertainty is limited to p-degrees of randomness [6, p.37], i.e. the unknown s.p. $u(x,t;\omega)$ depends on a finite number p of mutually independent r.v.'s $A_i(\omega)$, $i=1,\ldots,p$:

$$u\left(x,t;A_1(\omega),A_2(\omega),\cdots,A_p(\omega)\right)\,,\quad \omega\in\Omega. \tag{3}$$

Apart from the hypothesis (3) some technical conditions will be used in further developments:

$$0<d_1\leq d(\omega)\leq d_2\,,\;\; 0<\beta_1\leq\beta(\omega)\leq\beta_2\,,\;\; 0<a_1\leq a(\omega)\leq a_2\,,\;\; 0<b_1\leq b(\omega)\leq b_2\,, \tag{4}$$

for almost every (a.e.) $\omega\in\Omega$.

In this work we construct and compare numerical solutions for RPIDE model (1) by means both random methods: the random finite difference scheme (RFDS) and the random exponential time differencing (RETD). Both random methods were applied for local random problems, that is without integral term, in the recent papers [4] and [3]. We analyse the qualitative and computational properties such as the positivity, stability and consistency. Furthermore, due to the random framework requires not only the computation of the numerical solution but also its expectation, variance and storage accumulation problems we mix the sample approaches developed together Monte Carlo method. For the numerical convergence we use a Cauchy type condition.

2 The RFDS method

For a fixed horizon time $T\in\mathbb{R}$, let us consider a uniform partition of the temporal domain $[0,T]$, of the form $t^n=nk$, $0\leq n\leq N$ such that $Nk=T$. With respect to the spatial variable, let us define the step-size $\Delta x=h$ such that $\delta=hR$ for an arbitrary positive integer R. Note that initially we have a finite spatial domain $[-\alpha,\alpha]$, $\alpha>0$, so we built a partition $\mathfrak{D}^0$ having $2M+1$ equidistant nodes x_j of the form $x_j=jh$, $-M\leq j\leq M$, that is $\mathfrak{D}^0=\{x_{-M},x_{-M+1},\ldots,x_{-1},x_0,x_1,\ldots,x_{M-1},x_M\}$, such that M is the positive integer $M=\left\lceil\frac{\alpha}{h}\right\rceil$ with $\lceil\cdot\rceil$ denoting the ceiling function, thus $\frac{\alpha}{h}-1<M\leq\frac{\alpha}{h}$. In accordance with the spreading of the population due to the diffusive part of (1), we consider an extensible uniform spatial mesh $\mathfrak{D}^n$. Its number of nodes depends on each time level n in the sense that $\mathfrak{D}^n$ grows by adding to $\mathfrak{D}^{n-1}$ one more point on the left hand side and other on the right from $n=1$ until $n=N$. Note that these variable domains take the form $\mathfrak{D}^n=\{x_{-M-n},x_{-M-n+1},\ldots,x_{-1},x_0,x_1\ldots,x_{M+(n-1)},x_{M+n}\}$ for $0\leq n\leq N$. We denote the numerical approximation of the solution s.p. of the non-local RPIDE problem (1) by $u_j^n(\omega)\approx u(x_j,t^n;\omega)$, $-M-n\leq j\leq M+n$, $0\leq n\leq N$, $\omega\in\Omega$. Now in order to construct the RFDS, throughout the approximation of the m.s. derivatives by difference approximations, we approximate the m.s. time derivatives in (1) by a forward first-order approximation and by centred second-order approximation for the m.s. spatial derivatives. For the approximation of integral part of (1) let us consider previously the change of variables $z=x-y$ and take advantage of the compact support $[-\delta,\delta]$ of the kernel $\psi(x)$. We choose the random trapezoidal quadrature inspired in [5, p.51] with step-size h matching both differential and integral parts of the RFDS of RPIDE (1). The numerical integration of random convolution s.p. (2) considering $z=x-y$ at the node (x_j,t^n) denoting by $I_j^n(\omega)\approx(\psi*u)(x_j,t^n;\omega)$, takes the form

$$I_j^n(\omega)=\frac{h}{2}\psi_{-R}\,u_{j+R}^n(\omega)+h\sum_{s=-R+1}^{R-1}\psi_s\,u_{j-s}^n(\omega)+\frac{h}{2}\psi_R\,u_{j-R}^n(\omega)=h\sum_{s=-R+1}^{R-1}\psi_s\,u_{j-s}^n(\omega)\,,$$
$$-M-n\leq j\leq M+n\,,\;0\leq n\leq N,\omega\in\Omega\,, \tag{5}$$

where ψ_s denotes the value $\psi(x_s)$ of the kernel function at x_s, $\psi_{-R} = \psi_R = 0$. Figure 1 illustrates the successive addition of end points in the variable numerical domain for the first time levels up to $n = 2$.

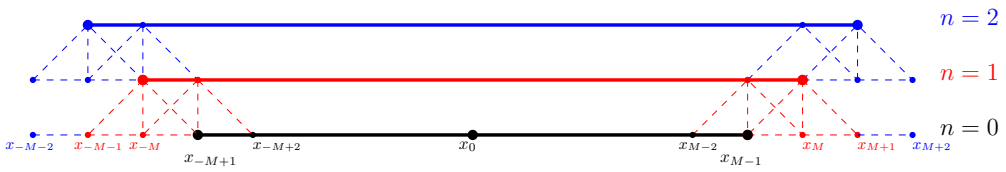


Figure 1: Spatial mesh growing over time levels n, $0 \le n \le 2$ with the non-zero support points.

Following the time process and using the induction argument over n, from the domain $\mathfrak{D}^n$ one gets the numerical solution s.p. for t^{n+1}, $0 \le n \le N-1$, in the numerical domain $\mathfrak{D}^{n+1}$ obtaining the following random explicit finite difference scheme for $-M-n+1 \le j \le M+n-1$, $0 \le n \le N-1$, $\omega \in \Omega$,

$$u_j^{n+1}(\omega) = \frac{k}{h^2} d(\omega) \left(u_{j-1}^n(\omega) + u_{j+1}^n(\omega)\right) + \left(1 - \frac{2k}{h^2} d(\omega)\right) u_j^n(\omega) + k\,\beta(\omega)\, u_j^n(\omega) \left(1 - a(\omega)\, u_j^n(\omega) - b(\omega)\, I_j^n(\omega)\right), \quad (6)$$

and

$$u_{-M-n}^{n+1}(\omega) = \frac{k}{h^2} d(\omega)\, u_{-M-n+1}^n(\omega), \quad 0 \le n \le N-1,\ \omega \in \Omega, \quad (7)$$

$$u_{M+n}^{n+1}(\omega) = \frac{k}{h^2} d(\omega)\, u_{M+n-1}^n(\omega), \quad 0 \le n \le N-1,\ \omega \in \Omega, \quad (8)$$

$$u_{-M-n-1}^{n+1}(\omega) = u_{M+n+1}^{n+1}(\omega) = 0, \quad 0 \le n \le N-1,\ \omega \in \Omega, \quad (9)$$

$$u_j^0(\omega) = f_j, \quad -M \le j \le M,\ \omega \in \Omega. \quad (10)$$

In the common case where both the kernel $\psi(x)$ and the initial density are symmetric, i.e. $\psi(-x) = \psi(x)$ and $f(-x) = f(x)$, see [1], it is easy to prove that numerical scheme (6)–(10) preserves the spatial symmetry for the random numerical solution at all time levels n, $0 \le n \le N$, that is, $u_{-j}^n(\omega) = u_j^n(\omega)$, $1 \le j \le M+n$, $\omega \in \Omega$.

Dealing with populations the **positivity** of the numerical solution needs to be preserved, then we explore sufficient conditions over the step-sizes (h, k) to guarantee the positivity as well as the **stability** of the numerical solution s.p. $u_j^n(\omega)$, $-M-n \le j \le M+n$, $0 \le n \le N$, $\omega \in \Omega$.

Definition 2.1 *The RFDS* (6)–(10) *is said to be m.s. stable in the fixed station sense in the domain X, with $X \subset \mathbb{R}$ and $T \in \mathbb{R}^+$, if for every partition with $k = \Delta t$, $h = \Delta x$ such that $Nk = T$ and $h = \delta R$ with $\delta \in \mathbb{R}^+$ and R an arbitrary positive integer, it is verified for each time level n, $0 \le n \le N$,*

$$\|u_j^n(\omega)\|_2 \le C, \qquad -M-n \le j \le M+n, \quad \omega \in \Omega,$$

where the constant $C > 0$ is independent of both the step-sizes (h, k) and the time level n.

The following Lemma 2.1 establishes the conditional positivity and boundness of the sample numerical solution and it provides a way to show the stability of the RFDS (6)–(10). This result exhibits conditionally as in the deterministic framework.

Lemma 2.1 *Let us assume conditions given by* (4) *and that the initial population density of the RPIDE* (1)–(2) *does not exceed the carrying capacity $0 \le f(x) \le \frac{1}{a_2}$, $x \in [-\alpha, \alpha]$, see Lemma 3.1*

and Theorem 3.2 [1]. Considering small enough step-sizes (h,k) *such that*

$$k \leq \frac{h^2}{2d_2+\rho h^2}\,, \quad \rho = \max\left\{\frac{2\beta_2 b_2}{a_1}, \beta_2\right\}, \tag{11}$$

then the sampling numerical solution $\{u_j^n(\omega)\}$ *for an arbitrary fixed event* $\omega \in \Omega$ *satisfies*

$$0 \leq u_j^n(\omega) \leq \frac{1}{a(\omega)}, \qquad -M-n \leq j \leq M+n,\ 0 \leq n \leq N.$$

In fact, under conditions of Lemma 2.1 it follows that for $-M-n \leq j \leq M+n$, $0 \leq n \leq N$,

$$\|u_j^n(\omega)\|_2 \;=\; \left(\mathbb{E}\left[\left(u_j^n(\omega)\right)^2\right]\right)^{1/2} = \left(\int_\Omega \left(u_j^n(\omega)\right)^2 f_{u_j^n(\omega)}(\omega)\,\mathrm{d}\omega\right)^{1/2} \leq \frac{1}{(a_1)^2}\,,$$

that is, the RFDS (6)–(10) is m.s. stable in the sense of Definition 2.1.

By extending the definition of the **consistency** of a RFDS with a random partial differential equation introduced in [4] to a RPIDE problem we provide the following definition.

Definition 2.2 *Let us consider a RFDS* $F(u_j^n(\omega))=0$, $\omega\in\Omega$, *for a RPIDE* $\mathcal{L}(U(\omega))=0$, $\omega\in\Omega$, *and let the local truncation error* $T(U_j^n(\omega))$ *for a fixed event* $\omega\in\Omega$ *be defined by*

$$T(U_j^n(\omega)) = F(U_j^n(\omega)) - \mathcal{L}(U_j^n(\omega)), \tag{12}$$

where $U_j^n(\omega) = U(x_j,t^n;\omega)$ *denotes the theoretical solution of the RPIDE evaluated at* (x_j,t^n) *for a fixed event* $\omega\in\Omega$. *The RFDS is said to be m.s. consistent with respect to the RPIDE if*

$$\|T(U_j^n(\omega))\|_2 \to 0 \;\; as \;\; h=\triangle x\to 0,\; k=\triangle t\to 0.$$

To prove the m.s. consistency of the RFDS (6)–(10) with the RPIDE (1)–(2) we fixed an event $\omega\in\Omega$. Then we use the hypothesis (3)–(4); assume that the exact solution $U(x,t;\omega)$ is four times continuously differentiable with respect to x and two times continuously differentiable with respect to t; the kernel $\psi(x)$, $x\in[-\delta,\delta]$, is two times continuously differentiable; we use the error of the trapezoidal rule of a finite Riemann integral, see [5, p.54]; and we develop the Taylor expansions of the partial derivatives involved in (12). Thus it follows the m.s. consistency.

3 The RETD method

We focus the construction of a RETD scheme in the symmetric case because the simulations in Section 4 concern this type of problems however the extension to the general case is straightforward. We use the same step-sizes discretization as in the symmetric case RFDS. Let us take $\Delta x = h = \frac{\delta}{R}$ and consider the numerical spatial domain $[0,(M+N)h]$ where the integers M and N are defined by $M=\left\lceil\frac{\alpha}{h}\right\rceil$ and $N=\frac{T}{k}$, respectively. The spatial nodes are $x_j=jh$, $0\leq j\leq M+N$. Taking centered finite difference approximations of the spatial second derivative in the RPIDE (1) with symmetric initial condition $f(x)$ and kernel $\psi(x)$ we achieve this semidiscretized random system of ODE's in time which can be rewritten in the following vector form

$$\left.\begin{array}{rcl} \dfrac{d}{dt}\mathbf{u}(t;\omega) &=& \mathbf{A}(\omega)\,\mathbf{u}(t;\omega)+\mathbf{b}(\mathbf{u}(t;\omega);\omega)\,, \quad 0<t<T,\ \omega\in\Omega\,, \\ \mathbf{u}(0;\omega) &=& [f(x_0),\ldots,f(x_{M-1}),0,\ldots,0]^T\,, \end{array}\right\} \tag{13}$$

where $\mathbf{u}(t;\omega) = [u_0(t;\omega), \cdots, u_{N+M-1}(t;\omega)]^T$, $0 < t < T$, $\omega \in \Omega$,

$$\mathbf{A}(\omega) = \frac{d(\omega)}{h^2} \begin{bmatrix} -2 & 2 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & & & 0 \\ 0 & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{bmatrix}, \tag{14}$$

is a random tridiagonal matrix of size $(M+N)\times(M+N)$ and the random vector

$$\mathbf{b}(\mathbf{u}(t;\omega);\omega) = [\beta(\omega)u_0(t;\omega)\,(1 - a(\omega)\,u_0(t;\omega) - b(\omega)\,I_0(t;\omega))\,, \ldots, \\ \beta(\omega)u_{M+N-1}(t;\omega)\,(1 - a(\omega)\,u_{M+N-1}(t;\omega) - b(\omega)\,I_{M+N-1}(t;\omega))]^T\,, \tag{15}$$

where $I_j(t;\omega) \approx (\psi * u)\,(x_j, t;\omega)$, $j = 0, \ldots, M+N$, denote the discrete integral expressions. The initial condition $f(x)$ is a deterministic function. Due to the r.v $d(\omega)$ verifies condition (4) that is, it is bounded, it is verifies that random matrix $\mathbf{A}(\omega) \in L_{2p}^{(M+N)\times(M+N)}(\Omega)$. Furthermore $\mathbf{A}(\omega)$ verifies that for its entries $a_{i,j}(\omega)$ there exist positive constants $s_{i,j}$, $\ell_{i,j}$ satisfying that its absolute moments of order r are bounded, that is,

$$\mathbb{E}\left[|a_{i,j}(\omega)|^r\right] \le s_{i,j}\,(\ell_{i,j})^r < +\infty\,, \quad \forall r \ge 0\,, \quad \forall i,j : 1 \le i,j \le M+N\,.$$

Furthermore, let us assume that the r.v.'s $\beta(\omega)$, $b(\omega)$ and $a(\omega)$ satisfy conditions (4). Applying [3, Prop. 2.7] for the time $[t^n, t^{n+1}]$, the random system of ODE's (13) taking $\mathbf{u}(t^n;\omega) \in L_{2p}^{(M+N)\times 1}(\Omega)$, $t^n = nk$, $0 \le n \le N-1$, it is equivalent to the random integral equation

$$\mathbf{u}(t^{n+1};\omega) = e^{\mathbf{A}(\omega)\,k}\,\mathbf{u}(t^n;\omega) + \int_0^k e^{\mathbf{A}(\omega)\,z}\,\mathbf{b}\left(\mathbf{u}(t^{n+1}-z;\omega);\omega\right)\mathrm{d}z\,. \tag{16}$$

Due the initial condition $f(x)$ is a deterministic function, the random solution $\mathbf{u}(t^{n+1};\omega)$ lies in $L_2^{(M+N)\times 1}(\Omega)$ for all $1 \le n \le N$ under conditions (4). To obtain a random numerical solution, $\mathbf{v}^{n+1}(\omega) \approx \mathbf{u}(t^{n+1};\omega)$, we approximate in (16) the random vector as follows $\mathbf{b}\left(\mathbf{u}(t^{n+1}-z;\omega);\omega\right) \approx \mathbf{b}\left(\mathbf{u}(t^n;\omega);\omega\right)$, by taking the known initial value corresponding to $z = k$. Then one gets

$$\mathbf{v}^{n+1}(\omega) \approx e^{\mathbf{A}(\omega)\,k}\,\mathbf{u}(t^n;\omega) + \left(\int_0^k e^{\mathbf{A}(\omega)\,z}\,\mathrm{d}z\right)\,\mathbf{b}\left(\mathbf{u}(t^n;\omega);\omega\right)\,, \quad 0 \le n \le N-1\,. \tag{17}$$

For a fixed event $\omega \in \Omega$, instead of computing the exact integration of the exponential matrix involving the tricky calculus of an inverse matrix we use the accurate Simpson quadrature rule [3, Sec. 2]. Now, we can transit from the sampling ETD scheme to the RETD scheme under conditions (4)

$$\left.\begin{array}{rcl} \mathbf{u}^{n+1}(\omega) & = & e^{\mathbf{A}(\omega)\,k}\,\mathbf{u}^n(\omega) + k\,\Phi(\mathbf{A}(\omega),k)\,\mathbf{b}\left(\mathbf{u}^n(\omega);\omega\right)\,, \quad 0 \le n \le N-1 \\ \mathbf{u}^0(\omega) & = & [f(x_0), \ldots, f(x_{M-1}), 0, \ldots, 0]^T \in \mathbb{R}^{N+M}, \end{array}\right\} \tag{18}$$

with

$$\mathbf{b}(\mathbf{u}^n(\omega);\omega) = [\beta(\omega)u_0^n(\omega)\,(1 - a(\omega)\,u_0^n(\omega) - b(\omega)\,I_0^n(\omega))\,, \ldots, \\ \beta(\omega)u_{M+N-1}^n(\omega)\left(1 - a(\omega)\,u_{M+N-1}^n(\omega) - b(\omega)\,I_{M+N-1}^n(\omega)\right)]^T\,, \\ 1 \le n \le N-1\,, \tag{19}$$

$$I_j^n(\omega) = h\left(\psi_0\,u_j^n(\omega) + \sum_{\ell=1}^{R-1}\psi_\ell\,\left(u_{j+\ell}^n(\omega) + u_{|j-\ell|}^n(\omega)\right)\right)\,, \quad 0 \le j \le M+N-1\,, \quad 0 \le n \le N-1, \tag{20}$$

and

$$\Phi(\mathbf{A}(\omega), k) = \frac{1}{6}\left(\mathbf{I} + 4e^{\mathbf{A}(\omega)\frac{k}{2}} + e^{\mathbf{A}(\omega)k}\right), \tag{21}$$

where $\mathbf{I}$ the identity matrix of size $(M+N)\times(M+N)$. Taking advantage of the fact that matrix $A(\omega)$ defined in (14) is a Metzler one, the use of properties concerning the exponential of Metzler matrices and the logarithmic norm allow to collect sufficient conditions in order to guarantee that the random population $\mathbf{u}^n(\omega)$ remains positive, not overcome the carrying capacity of the habitat and that the RETD scheme (18)–(21) is m.s. stable.

Theorem 3.1 *Let $d(\omega)$, $a(x)$, $b(\omega)$ and $\beta(x)$ be r.v.'s satisfying conditions* (4)*. Let us assume initial condition $f(x)$ be a positive continuous function compactly supported in the region $[-\alpha,\alpha]$ such that $f(-\alpha)=f(\alpha)=0$, and the kernel $\psi(x)$ be a non-negative continuous function compactly supported in the region $[-\delta,\delta]$ such that $\psi(-\delta)=\psi(\delta)=0$ verifying $\int_{-\delta}^{\delta}\psi(x)\,\mathrm{d}\,x=1$. Considering small enough step-sizes (h,k) satisfying condition* (11)*, then for each time level n, $0\le n\le N$,*

- *the numerical solution of the random problem* (1)–(4) *constructed by a sampling approach of the RETD scheme* (18)–(21) *satisfies $0\le u_j^n(\omega)\le\frac{1}{a(\omega)}$, for $0\le j\le M+N$ a.e. $\omega\in\Omega$.*
- *The RETD scheme* (18)–(21) *is m.s. stable in the fixed station sense.*

4 Simulations

In the random framework, discretization schemes as RFDS and RETD directly handled present computational troubles. These drawbacks are derived from the storage accumulation of symbolic expressions at the intermediate levels throughout iteration process, see [4, 3]. This fact motivates the use of the sampling scheme approach combined with Monte Carlo (MC) method to compute the mean and the standard deviation of the approximate numerical solution s.p. Computations have been carried out by Matlab© software version R2019b Update 3 for Windows 10Pro (64-bit) AMD Ryzen Threadripper 2990WX 32-Core Processor, 3.00 GHz. We have used parallel computing, by `parfor` commmad, for optimizing the code.

The goal of this random example is twofold: on one hand to illustrate the numerical convergence for the statistical moments of solution s.p. obtained by both methods as the time step k is refined; and on the other hand to show that RETD scheme (18)–(21) becomes less efficient that RFDS (6)–(10) with symmetry as the number of MC realizations K increases. In this example we take the following initial condition and kernel function, respectively,

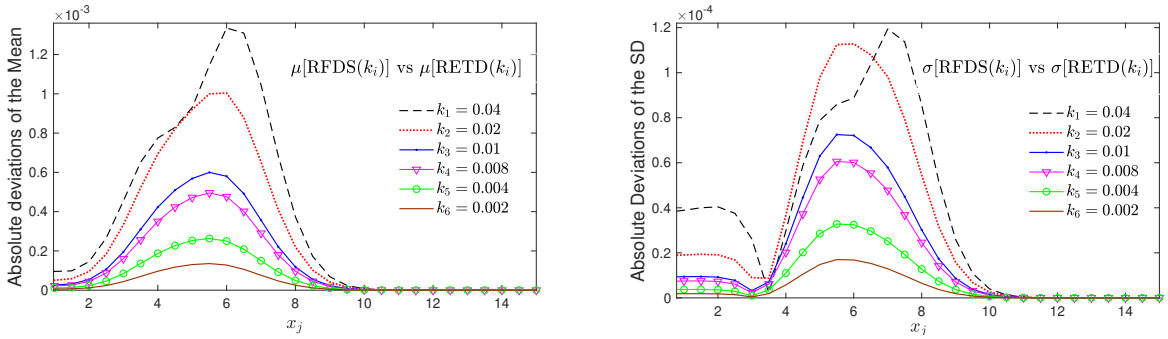
$$f(x)=\begin{cases}\frac{1}{50}(25-x^2), & |x|\le 5,\\ 0, & |x|>5,\end{cases}\qquad \psi(x)=\begin{cases}\frac{2}{9}\left(3-2|x|\right), & |x|\le 1.5,\\ 0, & |x|>1.5.\end{cases} \tag{22}$$

We consider that the r.v. $d(\omega)$ is a Gaussian distribution of mean $\mu=1$ and standard deviation $\sigma=0.1$ truncated on the interval $[0.8,1.2]$, i.e, $d(\omega)\sim \mathrm{N}_{[0.8,1.2]}(1,0.1)$. And we choose the rest of random distributions for the r.v.'s $\beta(\omega)$, $a(\omega)$ and $b(\omega)$ as follows $\beta(\omega)\sim \mathrm{Beta}_{[0.3,1]}(3,2)$, $a(\omega)\sim \mathrm{N}_{[0.6,1.4]}(1,0.2)$ and $b(\omega)\sim \mathrm{Gamma}_{[0.1,1]}(2,5)$. Thus, with this election of the r.v.'s involved in the random problem (1)-(2) it is obtained the following bounds

$$\left.\begin{array}{ll} 0<d_1=0.8\le d(\omega)\le d_2=1.2\,, & 0<\beta_1=0.3\le\beta(\omega)\le\beta_2=1\,,\\ 0<a_1=0.6\le a(\omega)\le a_2=1.4\,, & 0<b_1=0.1\le b(\omega)\le b_2=1\,.\end{array}\right\} \tag{23}$$

Figures 2-(a) and 2-(b) illustrate the decreasing behaviour of the absolute deviations between the first statistical moments (mean, μ, and standard deviation, σ) computed by both methods RFDS and RETD as the step-size $k\in\{0.04,0.02,0.01,0.008,0.004,0.002\}$ is refined. The time horizon

is $T = 0.8$ and the number of MC realizations is $K = 4000$. Taking the spatial step-size $h = 0.5$ fixed, each time step-size $\{k_i\}_{i=1}^{6}$ guarantees that the stability condition (11) is satisfied for the data (23). Although both methods generate close solutions, in the following experience we show



(a) (b)

Figure 2: Comparative graphics of the absolute deviations by RFDS and RETD methods as the time step-sizes $\{k_i\}$ decreases. Plot (a): for the mean μ. Plot (b): for the standard deviation σ.

the RETD scheme is not competitive computationally with RDFS for a long time T even for a few MC simulations K. Table 1 collects the timings in seconds computed using `cputime` function in Matlab© (CPU,s) as well as the real seconds elapsed by `tic,toc` commands. The approximate statistical moments of the solution s.p., (μ/σ), have been computed at time $T = 80$ with parameters $(h, k) = (0.5, 0.08)$ as the number of MC simulations $K \in \{25, 50, 100\}$ increases.

Table 1: CPU time seconds (CPU,s) and real time (in seconds) spent to compute the mean and the standard deviation (μ/σ) by means RFDS and RETD methods.

K	(μ/σ)[RFDS] CPU,s	(μ/σ)[RETD] CPU,s	(μ/σ)[RFDS] real time (s)	(μ/σ)[RETD] real time (s)
25	0.5781	100.3281	0.52	2052.04
50	0.8125	448.2188	0.76	4387.83
100	1.5938	605.2344	1.11	8942.53

Now we analyse the numerical convergence of the approximations to the mean and the standard deviation of the solution s.p. generated by RFDS (6)–(10) with symmetry. As the solution s.p. is not available the reliability of the numerical results have been checked using the concept of numerical convergence, that is, a Cauchy type condition varying the parameters (h, k) and the number of sampling realizations K. Firstly, we vary the number K of MC realizations with the pair (h, k) fixed as $(h, k) = (0.5, 0.08)$ satisfying stability condition (11). We introduce the following notation related to the numerical mean and standard deviation

$$\text{AbsDev}\left[\mu\left(u_{K_\ell K_{\ell+1}}(x_j, T; \omega)\right)\right] = \left|\mu\left(u_{K_{\ell+1}}(x_j, T; \omega)\right) - \mu\left(u_{K_\ell}(x_j, T; \omega)\right)\right|, \quad (24)$$

$$\text{AbsDev}\left[\sigma\left(u_{K_\ell K_{\ell+1}}(x_j, T; \omega)\right)\right] = \left|\sigma\left(u_{K_{\ell+1}}(x_j, T; \omega)\right) - \sigma\left(u_{K_\ell}(x_j, T; \omega)\right)\right|, \quad (25)$$

where K_ℓ and $K_{\ell+1}$ are two successive number of realizations and T is the time horizon. Table 2 collects the maximum values of the absolute deviations (24)–(25) that are getting closer as K increases at time horizon $T = 80$. In a second stage of the numerical convergence study of RFDS, we fix the number of MC realizations $K = 4000$, based on the experiences above in Table 2 where very close successive approximations were obtained, and we refine the step-sizes (h, k) according

Table 2: Maximum values of the absolute deviations (24)–(25) between two successive realizations $\{K_\ell, K_{\ell+1}\}$.

$\{K_\ell, K_{\ell+1}\}$	$\left\|\text{AbsDev}\left[\mu\left(u_{K_\ell K_{\ell+1}}(x_j, 80; \omega)\right)\right]\right\|_\infty$	$\left\|\text{AbsDev}\left[\sigma\left(u_{K_\ell K_{\ell+1}}(x_j, 80; \omega)\right)\right]\right\|_\infty$
$\{250, 1000\}$	$1.5201\text{e}-02$	$1.5139\text{e}-02$
$\{1000, 4000\}$	$1.2949\text{e}-02$	$1.0406\text{e}-02$
$\{4000, 16000\}$	$6.7019\text{e}-03$	$5.9791\text{e}-03$
$\{16000, 64000\}$	$3.7162\text{e}-03$	$3.6362\text{e}-03$

to the stability condition (11). Table 3 shows that the maximum values for the successive absolute deviations of the mean and the standard deviations, respectively, defined by

$$\text{AbsDev}\left[\mu\left(u_{K,h_\ell,h_{\ell+1}}(x_j, T; \omega)\right)\right] = \left|\mu\left(u_{K,h_{\ell+1}}(x_j, T; \omega)\right) - \mu\left(u_{K,h_\ell}(x_j, T; \omega)\right)\right|, \quad (26)$$

$$\text{AbsDev}\left[\sigma\left(u_{K,h_\ell,h_{\ell+1}}(x_j, T; \omega)\right)\right] = \left|\sigma\left(u_{K,h_{\ell+1}}(x_j, T; \omega)\right) - \sigma\left(u_{K,h_\ell}(x_j, T; \omega)\right)\right|, \quad (27)$$

have a decreasing behaviour as the pair (h_ℓ, k_ℓ) is refined at time horizon $T = 8$. These successive comparisons have been made in the common spatial domain $x_j = 0.5 \cdot j$, $0 \leq j \leq M + N = 110$.

Table 3: Maximum values of the absolute deviations (26)–(27) between successive approximations generated by RFDS.

$\{h_\ell, h_{\ell+1}\}$	$\left\|\text{AbsDev}\left[\mu\left(u_{K,h_\ell,h_{\ell+1}}(x_j, 8; \omega)\right)\right]\right\|_\infty$	$\left\|\text{AbsDev}\left[\sigma\left(u_{K,h_\ell,h_{\ell+1}}(x_j, 8; \omega)\right)\right]\right\|_\infty$
$\{0.5, 0.25\}$	$7.7921\text{e}-03$	$3.9410\text{e}-03$
$\{0.25, 0.125\}$	$5.5446\text{e}-03$	$8.0032\text{e}-04$
$\{0.125, 0.0625\}$	$3.2574\text{e}-03$	$6.9358\text{e}-04$

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