Probabilistic Analysis of Random Nonlinear Oscillators via the Equivalent Linearization Technique

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1 Introduction

Vibratory systems appear in Physics and Engineering when studying mechanical structures. In the nonlinear case, they can be formulated via second-order differential equations of the form [1],

$$\ddot{X}(t) + 2\beta \dot{X}(t) + \epsilon g(X(t)) + \omega_0^2 X(t) = Y(t), \quad t > 0, \tag{1}$$

where $X(t)$ is the spatial position, $Y(t)$ is the input or external forcing term acting upon the system. The model parameters describing the oscillatory system are the damping constant, $\beta > 0$, the undamped angular frequency, $\omega_0 > 0$, and the perturbation, ϵ, that affects to the nonlinear function $g(X(t))$, which depending on the spatial position. In real-world application the input may depend on external factors that are not not deterministically, but involving uncertainties. This motivates treating $Y(t)$ as a stochastic process rather than a deterministic function.

The analysis of random nonlinear oscillators subject to nonlinearities affected by a small perturbation can be approached by means of the perturbation technique [2]. This method allows us to obtain approximations of the main statistics of the steady-state of model (1), such as the mean, variance, and higher moments. Recently, in [3] a class of nonlinear oscillators whose restoring function is affected by small nonlinearities has been proposed and studied via the stochastic perturbation technique. In particular, the system is described by the nonlinear function $g(X(t)) = \sin(X(t))$ and excited by stationary zero-mean Gaussian stochastic processes. However, the restriction to problems with small perturbation parameter ($|\epsilon| \ll 1$) is the main limitation of the stochastic perturbation technique. An alternative method to construct reliable approximations of Equation (1) is the equivalent linearization technique [4].

Therefore, the aim of this contribution is to study a class of random nonlinear oscillators with a nonlinear function that depends only on the position via the stochastic equivalent linearization technique, in order to obtain approximations of the main statistics of the steady-state, including the first moments and the correlation function.

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2 Stochastic Equivalent Linearization

The stochastic linearization equivalent technique consists of, given a nonlinear differential equation,

$$\ddot{X}(t) + f\left(X(t), \dot{X}(t)\right) = Y(t), \quad t > 0, \tag{2}$$

to construct an equivalent linear differential equation,

$$\ddot{X}(t) + 2\beta_e \dot{X}(t) + \omega_e^2 X(t) = Y(t), \tag{3}$$

that it approximates the nonlinear model (2). To achieve this goal one minimizes the following error

$$N(t) = 2\beta_e \dot{X}(t) + \omega_e^2 X(t) - f\left(X(t), \dot{X}(t)\right), \tag{4}$$

in some sense.

In order to minimize (4), a common criterion is to consider the mean-square error, say, $N(t)$. Therefore, we impose that the parameters β_e and ω_e^2 be chosen such that

$$E\left\{N^2(t)\right\} = E\left\{\left[2\beta_e \dot{X} + \omega_e^2 X - f\left(X, \dot{X}\right)\right]^2\right\}, \tag{5}$$

is minimized for $t > 0$, where $E\{\cdot\}$ denotes the expectation operator.

2.1 Probabilistic Model Study

In the present study, we will assume that $g(t) = \sin(X(t))$, i.e., the nonlinear function g only depends on the position $X(t)$. This way

$$f\left(X(t), \dot{X}(t)\right) = 2\beta \dot{X}(t) + \omega_0^2 \left(X(t) + \epsilon \sin(X(t))\right). \tag{6}$$

It is important to mention that the transcendental function $\sin(X(t))$ can be represented as a truncation of its Taylor's series,

$$\sin(X(t)) \approx \sum_{m=0}^{M} \frac{(-1)^m}{(2m+1)!} (X(t))^{2m+1}. \tag{7}$$

In our subsequent analysis, we will consider $M = 2$, that corresponds to a Taylor's approximation of order 5,

$$\sin(X(t)) \approx X(t) - \frac{(X(t))^3}{3!} + \frac{(X(t))^5}{5!}. \tag{8}$$

Then, the values β_e and ω_e^2 are the solutions of the following equations:

$$\begin{aligned}
&2\left(\beta_e - \beta\right) E\left\{(\dot{X}(t))^2\right\} + \left(\omega_e^2 - \omega_0^2 - \epsilon\omega_0^2\right) E\left\{X(t)\dot{X(t)}\right\} + \frac{\epsilon\omega_0^2}{3!} E\left\{(X(t))^3 \dot{X}(t)\right\} \\
&\qquad - \frac{\epsilon\omega_0^2}{5!} E\left\{(X(t))^5 \dot{X}(t)\right\} = 0, \\
&\left(\omega_e^2 - \omega_0^2 - \epsilon\omega_0^2\right) E\left\{(X(t))^2\right\} + 2\left(\beta_e - \beta\right) E\left\{X(t)\dot{X}(t)\right\} + \frac{\epsilon\omega_0^2}{3!} E\left\{(X(t))^4\right\} \\
&\qquad - \frac{\epsilon\omega_0^2}{5!} E\left\{(X(t))^6\right\} = 0.
\end{aligned} \tag{9}$$

It can be proven that the stochastic process $X(t)$ is a stationary zero-mean Gaussian which is independent of $\dot{X}(t)$. Therefore, it can be deduced, from (9), that β_e and ω_e^2 are given by

$$\beta_e = \beta, \tag{10}$$

and

$$\begin{aligned} \omega_e^2 &= \omega_0^2 + \epsilon\omega_0^2 + \frac{-\frac{\epsilon\omega_0^2}{3!}E\left\{X^4\right\} + \frac{\epsilon\omega_0^2}{5!}E\left\{X^6\right\}}{E\left\{X^2\right\}} \\ &= \omega_0^2 + \epsilon\omega_0^2 - \frac{\epsilon\omega_0^2}{2}\left\{\sigma_X^2\right\} + \frac{\epsilon\omega_0^2}{8}\left\{\sigma_X^2\right\}^2. \end{aligned} \tag{11}$$

2.2 Variance Computation

Next, it is necessary the calculation of the variance in order to obtain the value of auxiliary parameters β_e and ω_e, thereby obtaining the solution of the equivalent linear equation (3) [5]. The variance σ_X^2 can be computed from the spectral density function, $S_{YY}(\omega)$, of the input stochastic process $Y(t)$ as follows

$$\sigma_X^2 = \int_0^\infty S_{XX}(\omega)\mathrm{d}\omega = \int_0^\infty |y(i\omega)|^2 S_{YY}(\omega)\mathrm{d}\omega, \tag{12}$$

where $y(i\omega)$ is the frequency response associated with the equivalent linear system described by Equation (3). It has the form

$$y(i\omega) = \frac{1}{\omega_e^2 - \omega^2 + 2i\beta_e\omega}. \tag{13}$$

To facilitate the computation of β_e and ω_e, we approximate the frequency response function $y(i\omega)$ by setting $\epsilon = 0$. Then $\beta_e = \beta$ and $\omega_e = \omega_0$, and $\sigma_{X_0}^2$ is given by

$$\sigma_{X_0}^2 = \int_0^\infty |y_0(i\omega)|^2 S_{YY}(\omega)\mathrm{d}\omega = \int_0^\infty \frac{1}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2} S_{YY}(\omega)\mathrm{d}\omega. \tag{14}$$

Finally, the relations between the power spectral density of $Y(t)$ and the correlation function Γ_{YY} is given by

$$S_{YY}(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty e^{-i\omega\tau}\Gamma_{YY}(\tau)\mathrm{d}\tau. \tag{15}$$

2.3 Steady-State Solution

It is important to mention that the analysis of the steady-state solution of the Equation (3) can be obtained by using the following representation of the solution of the linearized model,

$$X(t) = \int_0^\infty h(s)Y(t-s)\mathrm{d}s, \tag{16}$$

where

$$h(t) = \begin{cases} \left(\omega_e^2 - \beta_e^2\right)^{-\frac{1}{2}} e^{-\beta_e t}\sin\left(\left(\omega_e^2 - \beta_e^2\right)^{\frac{1}{2}} t\right), & \text{if } t > 0, \\ 0, & \text{if } t \le 0, \end{cases} \tag{17}$$

is the impulse response function for the underdamped case, $\dfrac{\beta_e^2}{\omega_e^2} < 1$.

On the other hand, it can be seen that $X(t)$ is a zero-mean Gaussian process, then its odd-order statistical moments are null,

$$E\{(X(t))^n\} = 0, \quad n = 1, 3, 5, \ldots \tag{18}$$

Additionally, it can be proven that the second-order moment for $X(t)$ is given by

$$E\left\{(X(t))^2\right\} = \int_0^\infty h(s) \int_0^\infty h(s_1)\Gamma_{YY}(s - s_1)\mathrm{d}s_1\mathrm{d}s. \tag{19}$$

From the previous consideration, $E\{X(t)\} = 0$, the covariance and correlation functions of $X(t)$ coincide, namely, $\mathbb{C}\{X(t_1)X(t_2)\} = \Gamma_{XX}(\tau)$, $\tau = |t_1 - t_2|$, where the correlation function of $X(t)$ is given by

$$\Gamma_{XX}(\tau) = \int_0^\infty h(s) \int_0^\infty h(s_1)\Gamma_{YY}(\tau - s_1 + s)\mathrm{d}s_1\mathrm{d}s. \tag{20}$$

2.4 Results and Discussion

To illustrate the previous theoretical results, we have chosen $Y(t) = \xi(t)$ a Gaussian white-noise process with zero-mean and correlation function $\Gamma_{YY}(\tau) = \pi S_0 \delta(\tau)$, where $\delta(\tau)$ is the Dirac delta function and $S_0 = \dfrac{1}{200\pi}$ is the noise power. In this case, Equation (1) becomes

$$\ddot{X}(t) + 2\beta_e \dot{X}(t) + \omega_e^2 X(t) = \xi(t). \tag{21}$$

For this model, we have considered the following parameters $\beta_e = \beta = 0.05$, $\omega_0^2 = 1$ and $\omega_e^2 = \omega_0^2 + \epsilon\omega_0^2 - \dfrac{\epsilon\omega_0^2}{2}\sigma_{X_0}^2 + \dfrac{\epsilon\omega_0^2}{8}(\sigma_{X_0}^2)^2$, where $\sigma_{X_0}^2 = 0.025$.

Remember that all the odd-order moments are null. Using expression (19), the second-order moment is determined by

$$E\left\{(X(t))^2\right\} = \frac{320}{12800 + 12641\epsilon}. \tag{22}$$

Now, applying (20), we obtain the following approximation of the correlation function,

$$\Gamma_{XX}(\tau) = \begin{cases} f_1(\tau) & \text{if } \tau \geq 0, \\ f_2(\tau) & \text{if } \tau < 0, \end{cases} \tag{23}$$

where

$$\begin{aligned} f_1(\tau) = {} & \frac{64}{(12768 + 12641\epsilon)(12800 + 12641\epsilon)} e^{-\frac{\tau}{20}} \left[(63840 + 63205\epsilon) \cos\left(\sqrt{\frac{399}{400} + \frac{12641\epsilon}{12800}} \right) \right. \\ & \left. + 20\sqrt{25536 + 25282\epsilon} \sin\left(\sqrt{\frac{399}{400} + \frac{12641\epsilon}{12800}} \right) \right], \end{aligned} \tag{24}$$

$$\begin{aligned} f_2(\tau) = {} & \frac{64}{(12768 + 12641\epsilon)(12800 + 12641\epsilon)} e^{\frac{\tau}{20}} \left[(63840 + 63205\epsilon) \cos\left(\sqrt{\frac{399}{400} + \frac{12641\epsilon}{12800}} \right) \right. \\ & \left. - 20\sqrt{25536 + 25282\epsilon} \sin\left(\sqrt{\frac{399}{400} + \frac{12641\epsilon}{12800}} \right) \right]. \end{aligned} \tag{25}$$

In Figure 1, we show the graphical representation of the correlation function, $\Gamma_{XX}(\tau)$, given by expression (23) for different values of ϵ.

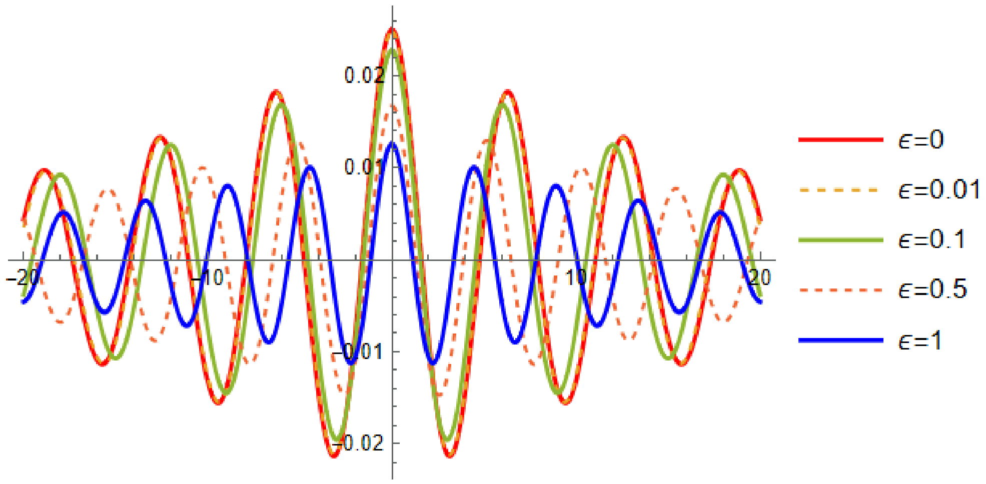

Figure 1: Correlation function $\Gamma_{XX}(\tau)$ of $X(t)$ for different values of ϵ.

3 Conclusions

In this paper, we have studied random nonlinear oscillators subject to perturbations on the nonlinear term that depends only on the position, that have been excited by a stationary zero-mean Gaussian process. In the study, we have obtained reliable approximations of the main statistics of the steady-state taking advantage of the stochastic equivalent linearization technique. Finally, we have illustrated the case in that the input excitation is a Gaussian white-noise. The numerical results preserve the properties of the statistics such as the symmetry with respect of the origin of the correlation function.

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