

Mean field and pair approximation analyses of a semiarid vegetation model

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1 Introduction

Water limitation in semiarid ecosystems results in discontinuous plant cover and complex vegetation dynamics, which can be prone to critical transitions between alternative stable states [1]. Many different models of semiarid vegetation have been proposed to analyse these dynamics [2], including spatially explicit discrete lattice models and continuous ones based on reaction-diffusion partial differential equations or including spatial kernels, as well as spatially implicit models in the form of systems of ordinary differential equations (ODE). In this communication we consider a well known spatially explicit discrete cellular automata semiarid vegetation model [3], which includes some of the main factors that are thought to be present in semiarid ecosystems, and we perform detailed analyses of the corresponding spatially implicit continuous models resulting from mean field [4] and pair approximations [5].

2 Methods

The original model in [3] consists in a lattice of cells or sites. Each cell can be in one of three possible states, vegetated (+), empty (0), and degraded (−), with transitions between them depending on the parameters that represent the basic factors affecting semiarid vegetation dynamics. Vegetated sites can become empty due to vegetation loss, with a constant mortality rate m . Empty sites can recover vegetation depending on a parameter representing plant establishment, b , but this recover can be negatively affected by competition, c . Empty sites can also be degraded at a constant rate d and degraded sites can recover its capacity to grow vegetation, becoming empty sites, at a base rate r , with this recovery being enhanced by local facilitation from neighbour vegetation, f . There is also a local dispersion effect in plant growth, but this aspect of the original model is lost in the spatially implicit approximations considered here.

2.1 Mean field model

In the mean field model approximation, there is no spatial variation and the system is characterized by the frequencies of the three different states. Thus, using the notation in [3], the dynamics of

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vegetated, ρ_+ , empty, ρ_0 , and degraded, ρ_- , frequencies, with $\rho_+, \rho_0, \rho_- \geq 0$ and $\rho_+ + \rho_0 + \rho_- = 1$, are given by the ODE system

$$\begin{aligned}\frac{d\rho_+}{dt} &= -m\rho_+ + \rho_0\rho_+(b - c\rho_+), \\ \frac{d\rho_0}{dt} &= m\rho_+ - \rho_0(\rho_+(b - c\rho_+) + d) + \rho_-(r + f\rho_+), \\ \frac{d\rho_-}{dt} &= d\rho_0 - \rho_-(r + f\rho_+).\end{aligned}$$

Writing $x \equiv \rho_+$ and $y \equiv \rho_-$, the system can be reduced to two independent equations,

$$\begin{aligned}\frac{dx(t)}{dt} &= -mx(t) + (1 - x(t) - y(t))x(t)(b - cx(t)), \\ \frac{dy(t)}{dt} &= d(1 - x(t) - y(t)) - y(t)(r + fx(t)),\end{aligned}\tag{1}$$

with $x, y \geq 0$, and $x + y \leq 1$.

2.2 Pair approximation model

The pair approximation model considers the frequencies of neighbouring pairs of sites ρ_{ij} , with $i, j \in \{+, 0, -\}$, with no asymmetry distinction, resulting in the system

$$\begin{aligned}\frac{d\rho_{++}}{dt} &= 2\rho_{+0}w_{\{0,+\}} - 2\rho_{++}w_{\{+,0\}}, \\ \frac{d\rho_{--}}{dt} &= 2\rho_{0-}w_{\{0,-\}} - 2\rho_{--}w_{\{-,0\}}, \\ \frac{d\rho_{+-}}{dt} &= \rho_{+0}w_{\{0,-\}} + \rho_{0-}w_{\{0,+\}} - \rho_{+-}(w_{\{-,0\}} + w_{\{+,0\}}), \\ \frac{d\rho_{00}}{dt} &= \rho_{+0}w_{\{+,0\}} + \rho_{0-}w_{\{-,0\}} - \rho_{00}(w_{\{0,-\}} + w_{\{0,+\}}), \\ \frac{d\rho_{+0}}{dt} &= \rho_{++}w_{\{+,0\}} + \rho_{+-}w_{\{-,0\}} + \rho_{00}w_{\{0,+\}} - \rho_{+0}(w_{\{0,+\}} + w_{\{0,-\}} + w_{\{+,0\}}), \\ \frac{d\rho_{0-}}{dt} &= \rho_{--}w_{\{-,0\}} + \rho_{+-}w_{\{+,0\}} + \rho_{00}w_{\{0,-\}} - \rho_{0-}(w_{\{0,-\}} + w_{\{0,+\}} + w_{\{-,0\}}),\end{aligned}\tag{2}$$

where $w_{\{ij\}}$ are the transition rates from state i to state j .

Since the sum of all frequencies total 1, the system can be reduced to 5 independent equations, which can be written as follows, involving three pair and two site frequencies,

$$\begin{aligned}\frac{d\rho_{++}}{dt} &= 2(\rho_+ - \rho_{++} - \rho_{+-})(\delta\rho_+ + \frac{1-\delta}{z} + \frac{(z-1)}{z}(1-\delta)\frac{\rho_+ - \rho_{++} - \rho_{+-}}{1 - \rho_+ - \rho_-})(b - c\rho_+) \\ &\quad - 2m\rho_{++}, \\ \frac{d\rho_{--}}{dt} &= 2d(\rho_- - \rho_{--} - \rho_{+-}) - 2\rho_{--}(r + \frac{z-1}{z}f\frac{\rho_{+-}}{\rho_-}), \\ \frac{d\rho_{+-}}{dt} &= (\rho_- - \rho_{--} - \rho_{+-})(\delta\rho_+ + \frac{(z-1)}{z}(1-\delta)\frac{\rho_+ - \rho_{++} - \rho_{+-}}{1 - \rho_+ - \rho_-})(b - c\rho_+) \\ &\quad + d(\rho_+ - \rho_{++} - \rho_{+-}) - \rho_{+-}(r + \frac{f}{z} + \frac{z-1}{z}f\frac{\rho_{+-}}{\rho_-} + m), \\ \frac{d\rho_+}{dt} &= (1 - \rho_+ - \rho_-)(\delta\rho_+ + (1-\delta)\frac{\rho_+ - \rho_{++} - \rho_{+-}}{1 - \rho_+ - \rho_-})(b - c\rho_+) - m\rho_+, \\ \frac{d\rho_-}{dt} &= d(1 - \rho_+ - \rho_-) - \rho_-(r + f\frac{\rho_{+-}}{\rho_-}),\end{aligned}\tag{3}$$

where z is the number of neighbours for each cell considered in the lattice model, either 4 or 8 depending on using in the cellular automata model Von Neumann or Moore neighbourhoods, respectively.

3 Results

We present in this communication a detailed analysis of the stability of the possible equilibria of both, mean field and pair approximation models, and bifurcation analyses in terms of the parameter b , representing environmental conditions, and its joint effects with the level of facilitation under different scenarios of mortality, competition and soil conditions. Examples of these results are shown in Figure 1, for the mean field model, and in Figure 2 for the pair approximation model.

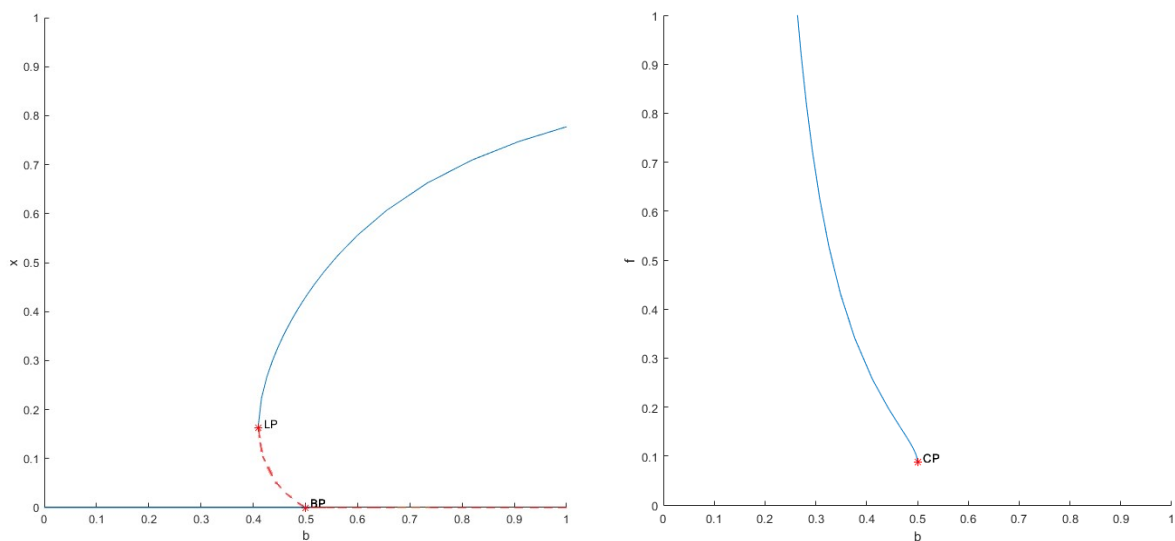


Figure 1: Left: Stable (blue) and unstable (dashed red) vegetation equilibria, and bifurcation points (LP and BP), as function of plant establishment b , in model (1). Right: Bifurcation analysis depending on parameters b and f .

We also analyse, for each model, the effect of different parameters and formulations on the values of a hydrological connectivity index, Flowlength [7], using the formulae for their expected values under random and aggregate vegetation distributions given in [8].

4 Conclusions

Basic analyses of the approximate models considered here were already presented in [3]. The more detailed analyses carried out in this work helps clarify the dependence of bistability regions and bifurcations on particular vegetation parameters and their joint effects. The comparison of mean field and pair approximation dynamics, in relation to the spatial cellular automata model, allows to separate the effects of local and full spatial aggregations on the dynamics of the system, and in particular on the expected values of connectivity indices like Flowlength, which can themselves produced a higher risk of critical transitions when incorporated into more complex models [6].

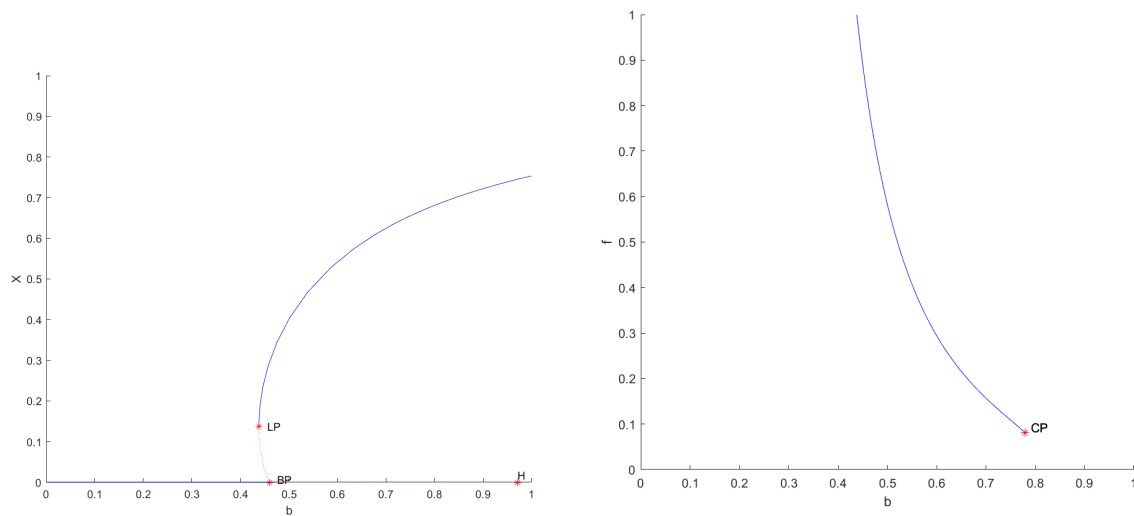


Figure 2: Left: Stable (blue) and unstable (dashed red) vegetation equilibria, and bifurcation points (LP, BP and H), as function of plant establishment b , in model (2). Right: Bifurcation analysis depending on parameters b and f .

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References

- [1] Rietkerk, M., Dekker, J.C., de Ruiter, P.C., and van de Koppel, J. 2004. Self-organized patchiness and catastrophic shifts in ecosystems. *Science* 305, 1926–1929.
- [2] Borgogno, F.; D’Odorico, P.; Laio, F.; Ridolfi, L. Mathematical models of vegetation pattern formation in ecohydrology. *Rev. Geophys.* **2009**, 47, RG1005.
- [3] Kefi, S., Rietkerk, M., van Baalen, M., and Loreau, M.: Local facilitation, bistability and transitions in arid ecosystems, *Theoretical Population Biology*, 71, 367–379, 2007.
- [4] Morozov, A., Poggiale, J.-C. From spatially explicit ecological models to mean-field dynamics: The state of the art and perspectives, *Ecol. Complex.* 10, 1–11, 2012.
- [5] Iwasa, Y. Lattice models and pair approximation in ecology. In Diekmann, O., Law, R., and Metz, J. A. J., editors, *The Geometry of Ecological Interactions. Simplifying Spatial Complexity*, pages 227–251. Cambridge University Press, Cambridge, 2000.
- [6] Mayor, Á.G.; Kéfi, S.; Bautista, S.; Rodríguez, F.; Cartení, F.; Rietkerk, M. Feedbacks between vegetation pattern and resource loss dramatically decrease ecosystem resilience and restoration potential in a simple dryland model. *Landsc. Ecol.* **2013**, 28, 931–942.

- [7] Mayor, Á.G., Bautista, S., Small, E.E., Dixon, M., Bellot, J. Measurement of the connectivity of runoff source areas as determined by vegetation pattern and topography. A tool for assessing potential water and soil losses in drylands. *Water Resour. Res.*, *44*, W10423, 2008.
- [8] Rodríguez, F., Mayor, Á.G., Rietkerk, M., Bautista, S. A null model for assessing the cover-independent role of bare soil connectivity as indicator of dryland functioning and dynamics. *Ecol. Indic.*, *94*, 512–519, 2018.