A control model for an indirect transmission disease

C. Coll, D. Ginestar, A. Herrero[1] and E. Sánchez

I.U. de Matemática Multidisciplinar, Universitat Politècnica de València
Camí de Vera s/n, Valencia, Spain.

1 Introduction

Some diseases, such as the salmonella and the brucellosis, are transmitted by contact of the susceptible individuals with the bacteria in the environment. To model this kind of diseases it is necessary to consider as state variables the susceptible individuals, $S(t)$, the infected ones, $I(t)$, and the contaminant, $C(t)$. Moreover, it is usually considered that the size of the population, $P = S(t) + I(t)$, remains constant for all $t \geq 0$, with the replacement of dead individuals represented in the model by $\mu(t)P$, for $\mu(t) \geq 0$. In such a way, the following discrete time system can be constructed

$$\begin{aligned} S(t+1) &= pS(t) - \sigma C(t)S(t) + \mu(t)P \\ I(t+1) &= qI(t) + \sigma C(t)S(t) \\ C(t+1) &= sC(t) + \beta I(t), t \geq 0; \end{aligned} \tag{1}$$

where $\sigma > 0$ represents the exposition rate of susceptible individuals, $0 < p,\ q < 1$ are the survival rates of the susceptible and infected population, respectively, and $0 < s < 1$ is the survival rate of the bacteria. Moreover, $\beta \geq 0$ represents the amount of contaminant or bacteria produced by each infected individual. This model is called an *indirect transmission epidemic model* and it is denoted by SIC from the name of the variables.

It is well known that the Basic Reproduction Number, R_0, of the model is a quantity that allows to determine the stability of the system to the disease-free equilibrium point. If $R_0 < 1$, the disease disappears and otherwise it remains [1]. In the unstable case different control strategies can be considered in order to eliminate or mitigate the spread of the disease[2, 3].

The main goal of this work is to analyze a control strategy based on a medical treatment on the infected individuals. We will consider a medicine that decreases the amount of bacteria produced by infected individuals. Under this consideration, we construct a new model and study different conditions for the parameters to achieve that the Basic Reproduction Number of the new model is less than one. We analyze the significance of the treatment effectiveness and the percentage of individuals who are treated. Finally, we show an optimal control problem for the percentage of treated infected individuals.

2 Control strategy

In this work, we apply a specific treatment to a percentag m of the infected individuals, where $0 < m < q$, and this treatment decreases the production of bacteria in this individuals. The

[1] aherrero@mat.upv.es

effectiveness of this medicine is given by the parameter $0 < \epsilon < 1$. So, we consider the infected individuals grouped in two new state variables: without medical treatment $I_1(t)$, and with medical treatment $I_2(t)$, that is $I(t) = I_1(t) + I_2(t)$. Under these considerations the new model can be written, for $t \geq 0$, as

$$\begin{aligned} S(t+1) &= pS(t) - \sigma C(t)S(t) + \mu(t)P \\ I_1(t+1) &= qI_1(t) + \sigma C(t)S(t) - mI_1(t) \\ I_2(t+1) &= qI_2(t) + mI_1(t) \\ C(t+1) &= sC(t) + \beta I_1(t) + (1-\epsilon)\beta I_2(t). \end{aligned} \tag{2}$$

As we have mentioned above, the population remains constant at any time, this implies that $\mu(t)P = (q-p)S(t) + (1-q)P$ and $S(t) = P - I_1(t) - I_2(t)$ for any $t \geq 0$. From this, the system can be reduced to

$$\begin{aligned} I_1(t+1) &= qI_1(t) + \sigma C(t)(P - I_1(t) - I_2(t)) - mI_1(t) \\ I_2(t+1) &= qI_2(t) + mI_1(t) \\ C(t+1) &= sC(t) + \beta I_1(t) + (1-\epsilon)\beta I_2(t). \end{aligned} \tag{3}$$

for $t \geq 0$. The linearization of this system around the disease-free equilibrium point is given by

$$\begin{pmatrix} I_1(t+1) \\ I_2(t+1) \\ C(t+1) \end{pmatrix} = \begin{pmatrix} q-m & 0 & \sigma P \\ m & q & 0 \\ \beta & (1-\epsilon)\beta & s \end{pmatrix} \begin{pmatrix} I_1(t) \\ I_2(t) \\ C(t) \end{pmatrix}. \tag{4}$$

In order to study the stability of the system, the state matrix can be decomposed as

$$E = \begin{pmatrix} q-m & 0 & \sigma P \\ m & q & 0 \\ \beta & (1-\epsilon)\beta & s \end{pmatrix} = \begin{pmatrix} q-m & 0 & 0 \\ m & q & 0 \\ 0 & 0 & s \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sigma P \\ 0 & 0 & 0 \\ \beta & (1-\epsilon)\beta & 0 \end{pmatrix} = T + F, \tag{5}$$

and the Basic Reproduction Number is given by the spectral radius of the matrix $F(I-T)^{-1}$ and will be denoted by $R_0(m, \alpha) = \rho(F(I-T)^{-1})$. A direct calculation leads to

$$R_0^2(m, \alpha) = \frac{\sigma\beta P}{(1-q)(1-s)} \left(1 - \frac{\epsilon\, m}{1-q+m}\right).$$

Clearly this Basic Reproduction Number is related to the corresponding one for the system without treatment, R_0, which has the form $R_0^2 = \dfrac{\sigma\beta P}{(1-q)(1-s)}$ as it is given in [3] and references given therein.

Starting from an unstable system without medication, $R_0 > 1$, we have that the system with the treatment will be stable if

$$R_0^2(m, \alpha) = R_0^2 \left(1 - \frac{\epsilon\, m}{1-q+m}\right) < 1 \Longleftrightarrow m \left(1 - \frac{1}{R_0^2} - \epsilon\right) < (1-q)\left(\frac{1}{R_0^2} - 1\right).$$

Imposing $\epsilon > 1 - \dfrac{1}{R_0^2}$, we obtain $R_0^2(m, \alpha) < 1 \Longleftrightarrow m > \dfrac{(1-q)(R_0^2-1)}{1-(1-\epsilon)R_0^2}$. Moreover, the hypothesis condition $m < q$ holds under this assumption.

The previous reasonign can be summarized in the following Proposition.

Proposition 1 *Let us consider the unstable SIC epidemic model given in (1) with $R_0 > 1$ and the medical treatment model given in (2). If the effectiveness ϵ satisfies $\epsilon > 1 - \dfrac{1}{R_0^2}$, then it is sufficient to apply the treatment to a percentage of infected individuals $m > \dfrac{(1-q)(R_0^2-1)}{1-(1-\epsilon)R_0^2}$ to ensure the stability of the medical treatment model to the disease-free equilibrium point.*

3 The optimal control problem

An optimal control problem involves adjusting the controls in a dynamic model so that we achieve a predetermined objective. The first thing we will have to do is to determine the type of controls we will use and the range where this controls can be chosen. Next, the construction of the objective function is an important issue since the optimal control results depend largely on its form. Generally, in the design of the objective function, the factors involved in the expression are weighted by balance coefficients based on their relative importance. Once defined this objective function, we assume that there is an optimal control and we compute it depending on the conditions that the control and the corresponding state variables have to meet.

In this section, we consider a medical treatment model where the percentage of infected individuals that are treated at time t is considered as a control variable $u(t)$:

$$\begin{aligned} I_1(t+1) &= qI_1(t) + \sigma C(t)(P - I_1(t) - I_2(t)) - u(t)I_1(t) \\ I_2(t+1) &= qI_2(t) + u(t)I_1(t) \\ C(t+1) &= sC(t) + \beta I_1(t) + (1-\epsilon)\beta I_2(t), t \geq 0, \end{aligned} \tag{6}$$

and we consider an objetive function of the form

$$J = y(T) + \sum_{t=0}^{T-1} \left(y(t) + \frac{a}{2} u^2(t) \right), \tag{7}$$

where $y(t) = c\ I_1(t) + d\ I_2(t) + e\ C(t),\ t = 0, \ldots, T$.

From some initial conditions, $I_1(0) = i_{10},\ I_2(0) = i_{20}$ and $C(0) = c_0$, and a final time T, our goal is to find the control sequence $\{u(j),\ j = 0, 1, \ldots, T-1\}$ to optimize the objective function (7) subject to the equations of the model (6). In order to do this, we apply the Pointryagin's Maximum Principle [4] and construct the Hamiltonian function H_t. For this problem, this function is defined by

$$\begin{aligned} H_t = y(t) + \frac{a}{2} u^2(t) &+ (qI_1(t) + \sigma C(t)(P - (1+u(t))I_1(t) - I_2(t)))\, \lambda_1(t+1) \\ &+ (qI_2(t) + u(t)I_1(t))\, \lambda_2(t+1) + (sC(t) + \beta(I_1(t) + (1-\epsilon)I_2(t))\, \lambda_3(t+1), \\ &t = 0, \ldots, T-1. \end{aligned} \tag{8}$$

Then, considering the system (6) and the initial conditions, $i_{10},\ i_{20}$, and c_0, the solution of the control problem will be given by the solution of the following optimality condition

$$\frac{\partial H_t}{\partial I_1(t)} = \lambda_1(t),\ \frac{\partial H_t}{\partial I_2(t)} = \lambda_2(t),\ \frac{\partial H_t}{\partial I_3(t)} = \lambda_3(t). \tag{9}$$

$$\frac{\partial H_t}{\partial u(t)} = a\, u(t) - I_1(t)(\lambda_1(t+1) - \lambda_2(t+1)) = 0, \tag{10}$$

and the conditions

$$0 = \frac{\partial H_t}{\partial \lambda_1(t)},\ 0 = \frac{\partial H_t}{\partial \lambda_2(t)},\ 0 = \frac{\partial H_t}{\partial \lambda_3(t)}.$$

Moreover, since the control variable $u(t)$ represents the percentage of infected individuals that are treated, there are some constraints imposed by the biological sense of this variable. That is, the sequence of optimal control has to satisfy that it is below the quantity

$$u^\star = \min\left\{1, \max\left\{0, \frac{I_1(t)}{a}(\lambda_1(t+1) - \lambda_2(t+1))\right\}\right\}$$

at any time.

However, to solve analytically the above equations is not a trivial problem in the sense that the system must be solved using a progressive part and a regressive one. Furthermore, the number of unknowns and the relations between them, imposed by the nature of the problem, make it difficult to obtain an explicit expression of the solution. Therefore, we carry out a numerical simulation to study the solution that provides us a good control.

4 Conclusions

Diseases due to indirect infections are closely related to the proliferation of bacteria or other elements that occur in the environment. Any large-scale bacterial spread would be disastrous due to the impact of epidemics on the health system and global economies. Our interest is to seek resources to stop this production through a control strategy such as medical treatments that reduce the production of bacteria. In this work, this strategy has been used to investigate under what conditions it is possible to stop the proliferation of the disease. Furthermore, taking into account the possible impact of the treatment, an optimal control problem has been presented that allows us to improve this impact in the direction of interest.

References

[1] L.J. Allen, P. van den Driessche, The basic reproduction number in some discrete-time epidemic models, *Journal of difference equations and applications*, 14(10-11): 1127-1147, 2008.

[2] B. Buonomo, D, Lacitignola, C. Vargas-De León, Qualitative analysis and optimal control of an epidemic model with vaccination and treatment, *Mathematics Computers in Simulation*, 100: 88-102, https://dx.doi.org/10.1016/j.matcom2013.11.005, 2014.

[3] B. Cantó, C. Coll, M.J. Pagán, J. Poveda, E. Sánchez, *A Mathematical Model to Control the Prevalence of a Directly and Indirectly Transmitted Disease, Mathematics*, 9(20): 2562; https://doi.org/10.3390/math9202562, 2021.

[4] L.S. Pontryagin, V.G. Boltyanskii, et all, The mathematical theory of optimal processes, Wiley, New York, 1962.