Parametric case of family study

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1 Introduction

Analysis of an scalar iterative expression of a Family of Methods, in order to determine its extensible to equation systems and its convergence rate.

Furthermore, we will study its behaviour when applied to a particular equation system, and it will be objective of our investigation to settle the stability of the Family of Methods, within some other aspects.

The expression under consideration it's presented like this (considering x_0 the initial estimation):

$$\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, k = 0, 1, 2, \ldots \\
z_k &= y_k - \frac{1}{\beta}\frac{f(y_k)}{f'(x_k)}, \\
x_{k+1} &= z_k - \frac{\left(2 - \frac{1}{\beta} - \beta\right) f(y_k) + \beta f(z_k)}{f'(x_k)}.
\end{aligned}$$

being β a non cero real or complex parameter.

At first glance, we can observe the method is naturally extensible to equation systems, due to the fact there are non other than derivative functions at denominators. This tendency allows us to simply swap them for the Jacobian $F'\left(x^{(k)}\right)$ matrix (obviously referring to the inverse of it), and setting them ahead of the rest of each expression. The commented conversion is presented as shown:

$$\begin{aligned}
y^{(k)} &= x^{(k)} - \left[F'\left(x^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right), k = 0, 1, 2, \ldots \\
z^{(k)} &= y^{(k)} - \frac{1}{\beta}\left[F'\left(x^{(k)}\right)\right]^{-1} F\left(y^{(k)}\right), \\
x^{(k+1)} &= z^{(k)} - \left[F'\left(x^{(k)}\right)\right]^{-1}\left(\left(2 - \frac{1}{\beta} - \beta\right)F\left(y^{(k)}\right) + \beta F\left(z^{(k)}\right)\right).
\end{aligned}$$

whenever $F'\left(x^{(k)}\right)$ is invertible.

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2 Analysis of the Method Family

2.1 Convergence rate

The error equation of the described method, in the scalar form, can be obtained using Taylor Series:

$$e_{k+1} = c_2^3 \frac{5\beta - 1}{\beta} e_k^4 + \frac{(10 - 36\beta)c_2^4 + 4(-1 + 6\beta)c_2^2 c_3}{\beta} e_k^5 + \mathcal{O}\left(e_k^6\right)$$

being $e_k = x_k - \alpha$, $c_j := \dfrac{f^{(j)}(\alpha)}{f'(\alpha)}$. The convergence rate is directly affected by the free parameter (5^{th} order if $\beta = \dfrac{1}{5}$, 4^{th} order otherwise).

$\square$

Similarly, vectorial convergence rate can be obtained making some rearrangements, getting as a result the following expression:

$$e_{k+1} = \Big(C_2^3(-\gamma - \delta + 5) + \beta C_2^2(1 + \gamma - \gamma^2) + 2C_2\delta\Big) e_k^4 + \mathcal{O}\left(e_k^5\right)$$

Where γ and δ are β-depending parameters. As we see, the vectorial method is 4^{th} order.

2.2 Efficiency index

We have calculated two types of index, one based on the one defined by Ostrowski and another defined by Cordero and Torregrosa (computational index) which is based on Ostrowski but that also takes into account the resolution of the systems involved in the method

Which are defined as follows:

$$OI = p^{\frac{1}{d}}, \text{ where } d \text{ represents the functional evaluations and } p \text{ the order of convergence}$$

$$CI = p^{\frac{1}{d+o}}, \text{ where } d \text{ and } p \text{ represents the same, but } o \text{ represents the number of products/quotients}$$

In our family the number of products/quotients is given by looking at the resolutions of the systems of equations using the Gauss-Jordan method.

So with all we obtain that

$$d = n^2 + 3n \text{ and } o = \frac{1}{3}n^3 + 3n^2 - \frac{1}{3}n$$

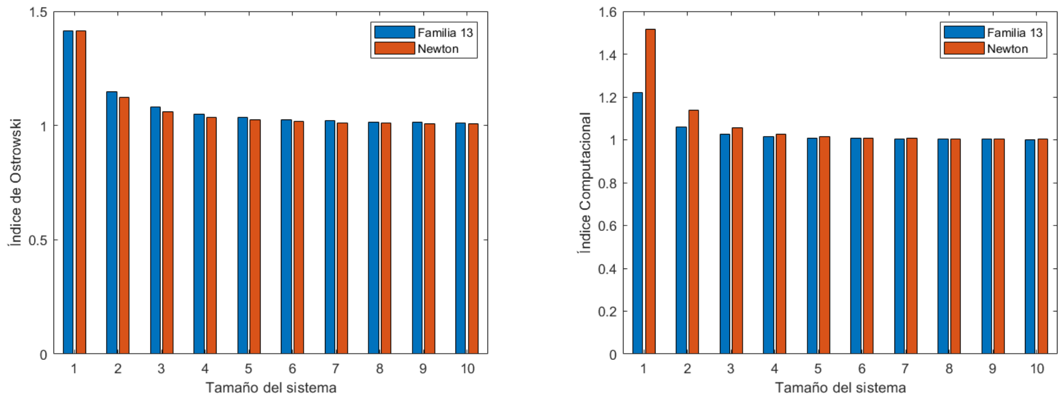
where n is the numbers of equations.

so far

$$OI = 4^{\frac{1}{n^2+3n}}$$

and

$$CI = 4^{\frac{1}{\frac{1}{3}n^3 + 4n^2 + \frac{8}{3}n}}$$



(a) Ostrowski index

(b) Coputational index

3 Results

We see that the family has the same order of convergence in the scalar and vectorial systems always that $\beta \neq \dfrac{1}{5}$, in that case the scalar case has one more order of convergence.

In terms of efficiency, the method can be compared to Newton's method extended to systems of equations in the next figures.

As we can see, the parametric family exhibits greater efficiency than the Newton method when comparing the Ostrowski index, but the opposite occurs when comparing the computational index.

Hence, we can conclude that it's not a bad method, but considering the computational index, which also takes into account the number of operations, a crucial aspect in large-scale systems, it appears that employing Newton's method would be more advantageous.

4 Estability of the Family

With the aim of studying stability overall, we apply to our given family the general second degree polynomial $p(x) = (x-a)\cdot(x-b)$, just to brush it over with de *Möbius* transformation, getting this as a result:

$$R(x,\beta) = x^4 \frac{-1+5\beta+14x\beta+14x^2\beta+6x^3\beta+x^4\beta}{-x^4+\beta+6x\beta+14x^2\beta+14x^3\beta+5x^4\beta}$$

This expression is our key to work, and so we calculate its critical points and proceed to plot the "basins".

The critical points are:

- $x = 0$
- $x = 1$
- The six roots of the polynomial $\beta + 7\beta z + 21\beta z^2 + (1+30\beta)z^3 + 21\beta z^4 + 7\beta z^5 + \beta z^6$, ex_i with $i = 1, \ldots, 6$ from now on.

Besides, infinity is another critical point, because $\lim\limits_{x\to 0} \dfrac{1}{R'\left(\frac{1}{x},\beta\right)} = 0$.

Plotting is done by evaluating the critical points in the derivative of the expression, giving us information of their behaviour.

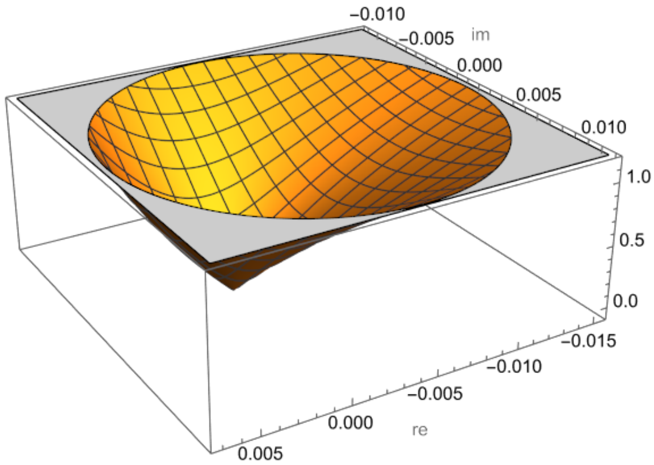


Figure 1: Representation of the $x = 1$ basin

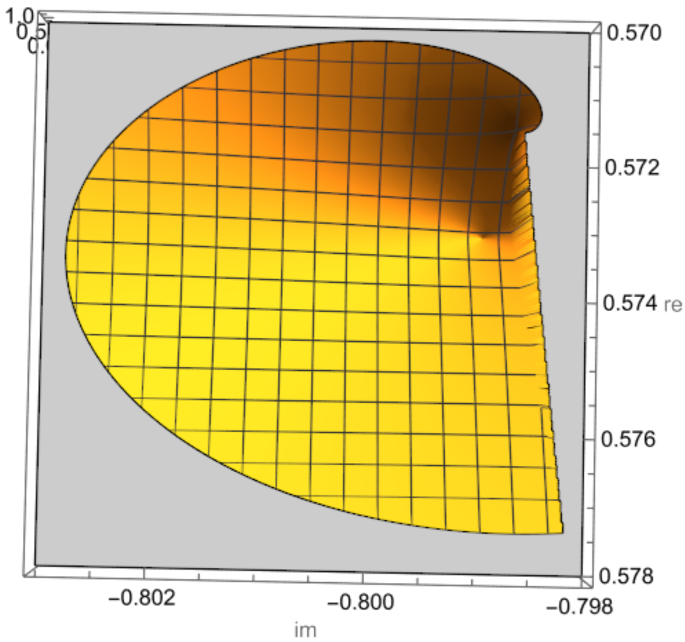


Figure 2: Representation of the $x = ex_3$ basin

Acknowledgements

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References

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